(Re)Creating an Environment of Group Mathematical Discovery

Todd CadwalladerOlsker

California State University, Fullerton

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Why this talk?

This talk is based on a class session that occurred in "Algebra and Probability for the Secondary Teacher" in Spring 2011. In this class session, a class exercise on polynomial functions veered into a discussion of horizontal transformations in a very illuminating way.

Since then, I've been thinking about what classroom factors led to this discussion and how I can encourage those factors in my own classroom. This talk is the result.

Given $n + 1$ points on the plane with different x-coordinates, we can generally find a polynomial function of degree n or less that goes through those points.

For example, two points determine a line (linear function), three points determine a parabola (quadratic function), etc.

The "Standard" Method

Given points $(x_0, y_0), (x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots, (x_n, y_n)$ with distinct x-coordinates, there is a polynomial

$$y = p_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

We can find the coefficients of the polynomial by solving the system of equations

$$\begin{aligned}
y_0 &= a_n(x_0)^n + a_{n-1}(x_0)^{n-1} + \cdots + a_1(x_0) + a_0 \\
y_1 &= a_n(x_1)^n + a_{n-1}(x_1)^{n-1} + \cdots + a_1(x_1) + a_0 \\
y_2 &= a_n(x_2)^n + a_{n-1}(x_2)^{n-1} + \cdots + a_1(x_2) + a_0 \\
&\vdots \\
y_n &= a_n(x_n)^n + a_{n-1}(x_n)^{n-1} + \cdots + a_1(x_n) + a_0
\end{aligned}$$

EXAMPLE

If we are given the points (-1,3),(0,-1),(1,2), and (2,-2), we can obtain the polynomial

$$y = a_3x^3 + a_2x^2 + a_1x + a_0$$

by solving for the coefficients in the system

$$\begin{aligned}
3 &= a_3(-1)^3 + a_2(-1)^2 + a_1(-1) + a_0 \\
-1 &= a_3(0)^3 + a_2(0)^2 + a_1(0) + a_0 \\
2 &= a_3(1)^3 + a_2(1)^2 + a_1(1) + a_0 \\
-2 &= a_3(2)^3 + a_2(2)^2 + a_1(2) + a_0
\end{aligned}$$

Example

Solving gives us

$$a_0 = -1,\ a_1 = \frac{11}{6},\ a_2 = \frac{7}{2},\ a_3 = \frac{-7}{3}$$

and therefore the polynomial

$$y = \frac{-7}{3}x^3 + \frac{7}{2}x^2 + \frac{11}{6}x - 1$$

fits all four of these points.

Lagrangian Interpolation

Solving a large system of equations is not always easy. Fortunately, there are other ways as well. Another way to find a polynomial that matches a set of data is *Lagrangian Interpolation.*

Let's again use the points (-1,3),(0,-1),(1,2), and (2,-2). The idea is that we can define $f(x)$ as the third-degree polynomial:

$$\begin{aligned} f(x) = {} & A(x-(^{-}1))(x-0)(x-1) \\ & + B(x-(^{-}1))(x-0)(x-2) \\ & + C(x-(^{-}1))(x-1)(x-2) \\ & + D(x-0)(x-1)(x-2) \end{aligned}$$

Example

In each line, one of the four factors $(x -^{-} 1), (x - 0), (x - 1),$ or $(x - 2)$ is missing. So, if we plug in the fact that $f(-1) = 3$, the first three lines become 0, and the only term to survive is

$$f(-1) = 3 = D((-1) - 0)((-1) - 1)((-1) - 2)$$

which gives us $D = \frac{-1}{2}$. Similarly, using (0,-1),

$$f(0) = -1 = C(0 - (-1))(0 - 1)(0 - 2)$$

which gives us $C = \frac{-1}{2}$. Continuing is this way, we get $B = -1$ and $A = \frac{-1}{3}$.

Example

Therefore, we have

$$\begin{aligned} f(x) = \frac{-1}{3}&(x - (^{-}1))(x - 0)(x - 1) \\ &+ -1(x - (^{-}1))(x - 0)(x - 2) \\ &+ \frac{-1}{2}(x - (^{-}1))(x - 1)(x - 2) \\ &+ \frac{-1}{2}(x - 0)(x - 1)(x - 2) \end{aligned}$$

which simplifies to

$$f(x) = \frac{-7}{3}x^3 + \frac{7}{2}x^2 + \frac{11}{6}x - 1$$

just as before.

Newton's Difference Method

If our points happen have particularly nice x-coordinates, namely $0, 1, 2, \ldots,$

$$(0, y_0), (1, y_1), (2, y_2), \ldots, (n, y_n)$$

we can use a very easy method known as Newton's Difference Method.

Example

Say we are given the points (0,3),(1,1),(2,5),(3,7). We look at the *forward differences*:

$$\Delta y_x = y_{x+1} - y_x$$

$$\Delta^2 y_x = \Delta y_{x+1} - \Delta y_x$$

and so on. In our example, we have

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	3	-2	6	-8
1	1	4	-2	
2	5	2		
3	7			

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	3	-2	6	-8
1	1	4	-2	
2	5	2		
3	7			

Now, notice that

$$
\begin{aligned}
y_0 &= 3 = \color{red}{1} \cdot 3 \\
y_1 &= 1 = \color{red}{1} \cdot 3 + \color{red}{1} \cdot -2 \\
y_2 &= 5 = 1 \cdot 1 + 1 \cdot 4 \\
&= (3 + -2) + (-2 + 6) \\
&= (\color{red}{1} \cdot 3) + (\color{red}{2} \cdot -2) + (\color{red}{1} \cdot 6) \\
y_3 &= 7 = 1 \cdot 5 + 1 \cdot 2 \\
&= (1 + 4) + (4 + -2) \\
&= (3 + -2) + (-2 + 6) + (-2 + 6) + (6 + -8) \\
&= (\color{red}{1} \cdot 3) + (\color{red}{3} \cdot -2) + (\color{red}{3} \cdot 6) + (\color{red}{1} \cdot -8)
\end{aligned}
$$

Pascal's Triangle

$$
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 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

$$\binom{x}{k} = \frac{x!}{k!(x-k)!}$$

If $x < k$, we define $\binom{x}{k} = 0$.

To get a function that fits out data points, we can write

$$y_0 = \binom{0}{0}(3) + \binom{0}{1}(-2) + \binom{0}{2}(6) + \binom{0}{3}(-8)$$

$$y_1 = \binom{1}{0}(3) + \binom{1}{1}(-2) + \binom{1}{2}(6) + \binom{1}{3}(-8)$$

$$y_2 = \binom{2}{0}(3) + \binom{2}{1}(-2) + \binom{2}{2}(6) + \binom{2}{3}(-8)$$

$$y_3 = \binom{3}{0}(3) + \binom{3}{1}(-2) + \binom{3}{2}(6) + \binom{3}{3}(-8)$$

$$f(x) = \binom{x}{0}(3) + \binom{x}{1}(-2) + \binom{x}{2}(6) + \binom{x}{3}(-8)$$

Binomial Form of Polynomials

We can define a "binomial form" polynomial as

$$\begin{aligned}\binom{x}{k} &= \frac{x!}{k!(x-k)!} \\ &= \frac{x(x-1)(x-2)\cdots(x-k+1)(x-k)\cdots(2)(1)}{k![(x-k)\cdots(2)(1)]} \\ &= \frac{x(x-1)(x-2)\cdots(x-k+1)}{k!}\end{aligned}$$

$\binom{x}{k}$ is a kth degree polynomial. For example,

$$\binom{x}{3} = \frac{x(x-1)(x-2)}{3!} = \frac{x^3}{6} - \frac{x^2}{2} + \frac{x}{3}$$

Example

Therefore, our function

$$f(x) = \binom{x}{0}(3) + \binom{x}{1}(-2) + \binom{x}{2}(6) + \binom{x}{3}(-8)$$

$$= 1(3) + \frac{x}{1!}(-2) + \frac{x(x-1)}{2!}(6) + \frac{x(x-1)(x-2)}{3!}(-8)$$

$$= -\frac{4}{3}x^3 + 7x^2 - \frac{23}{3}x + 3$$

In summary, Newton's Difference Method gives us a really nice way of developing a formula that matches data. Given the difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	y_0	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$
1	y_1	Δy_1	$\Delta^2 y_1$	
2	y_2	Δy_2		
3	y_3			

We can find a polynomial function

$$f(x) = \binom{x}{0}(y_0) + \binom{x}{1}(\Delta y_0) + \binom{x}{2}(\Delta^2 y_0) + \binom{x}{3}(\Delta^3 y_0)$$

In one of the classes I teach, "Algebra and Probability for the Secondary Teacher," I assigned the following problem in class:

> *Find a polynomial function that fits the following points, representing the population of Seattle (in thousands) in different years:*

$$(1950, 468), (1960, 557), (1970, 531)$$

$$(1980, 494), (1990, 516), (2000, 537)$$

(From Mathematics for the High School Teacher: An Advanced Perspective, by Usiskin et al)

I assigned students to work on this problem in groups of two or three. The intention of the problem was to have them use Lagrangian Interpolation to find a polynomial, and that's what most of the students did.

First, they changed the *x*-values to 50,60,70,80,90, and 100:

$$(50, 468), (60, 557), (70, 531)$$

$$(80, 494), (90, 516), (100, 537)$$

Lagrangian Interpolation

$$\begin{aligned} f(x) = {} & A(x-50)(x-60)(x-70)(x-80)(x-90) \\ & + B(x-50)(x-60)(x-70)(x-80)(x-100) \\ & + C(x-50)(x-60)(x-70)(x-90)(x-100) \\ & + D(x-50)(x-60)(x-80)(x-90)(x-100) \\ & + E(x-50)(x-70)(x-80)(x-90)(x-100) \\ & + F(x-60)(x-70)(x-80)(x-90)(x-100) \end{aligned}$$

Lagrangian Interpolation

$$
\begin{aligned}
f(50) &= 468 = F(-10)(-20)(-30)(-40)(-50) \\
f(60) &= 557 = E(10)(-10)(-20)(-30)(-40) \\
f(70) &= 531 = D(20)(10)(-10)(-20)(-30) \\
f(80) &= 494 = C(30)(20)(10)(-10)(-20) \\
f(90) &= 516 = B(40)(30)(20)(10)(-10) \\
f(100) &= 537 = A(50)(40)(30)(20)(10)
\end{aligned}
$$

Lagrangian Interpolation

$$F = \frac{468}{-12000000}$$
$$E = \frac{557}{2400000}$$
$$D = \frac{531}{-1200000}$$
$$C = \frac{494}{1200000}$$
$$B = \frac{516}{-2400000}$$
$$A = \frac{537}{12000000}$$

Lagrangian Interpolation

$$
\begin{aligned}
f(x) = {} & \frac{537}{12000000}(x-50)(x-60)(x-70)(x-80)(x-90) \\
& + \frac{516}{-2400000}(x-50)(x-60)(x-70)(x-80)(x-100) \\
& + \frac{494}{1200000}(x-50)(x-60)(x-70)(x-90)(x-100) \\
& + \frac{531}{-1200000}(x-50)(x-60)(x-80)(x-90)(x-100) \\
& + \frac{557}{2400000}(x-50)(x-70)(x-80)(x-90)(x-100) \\
& + \frac{468}{-12000000}(x-60)(x-70)(x-80)(x-90)(x-100)
\end{aligned}
$$

Solution 1

Simplifying, we get

$$f(x) = \frac{-x^5}{125000} + \frac{319x^4}{120000} - \frac{2003x^3}{6000} + \frac{23219x^2}{1200} - \frac{73993x}{150} + 4374$$

To check their work, the students calculated that

$$\begin{aligned} f(50) &= 468 \\ f(60) &= 557 \\ f(70) &= 531 \\ f(80) &= 494 \\ f(90) &= 516 \\ f(100) &= 537 \end{aligned}$$

However, one group of particularly bright students realized that because the x-values in the given points were equally spaced, they could change the x-values to 0,1,2,3,4, and 5, and use Newton's Difference Method.

$$(0, 468), (1, 557), (2, 531)$$
$$(3, 494), (4, 516), (5, 537)$$

Newton's Difference Method

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0	468	89	-115	104	-34	-96
1	557	-26	-11	70	-130	
2	531	-37	59	-60		
3	494	22	-1			
4	516	21				
5	537					

Solution 2

This gives us the polynomial

$$g(x) = 468\binom{x}{0} + 89\binom{x}{1} - 115\binom{x}{2} + 104\binom{x}{3} - 34\binom{x}{4} - 96\binom{x}{5}$$

which simplifies to

$$g(x) = \frac{-4x^5}{5} + \frac{79x^4}{12} - \frac{13x^3}{6} - \frac{1021x^2}{12} + \frac{2557x}{15} + 468$$

Solution 2

Again, I had the students check their work:

$$g(0) = 468$$
$$g(1) = 557$$
$$g(2) = 531$$
$$g(3) = 494$$
$$g(4) = 516$$
$$g(5) = 537$$

COMPARING SOLUTIONS

In class, I had students present their solutions at the board for a class discussion.

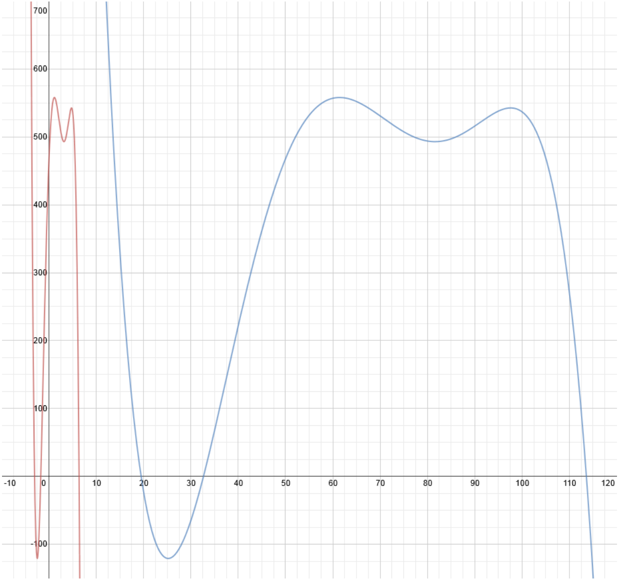
At first glance, the solutions look very different:

$$f(x) = \frac{-x^5}{125000} + \frac{319x^4}{120000} - \frac{2003x^3}{6000} + \frac{23219x^2}{1200} - \frac{73993x}{150} + 4374$$

$$g(x) = \frac{-4x^5}{5} + \frac{79x^4}{12} - \frac{13x^3}{6} - \frac{1021x^2}{12} + \frac{2557x}{15} + 468$$

Comparing Solutions

To get a sense of what was happening, we looked at graphs of the two functions: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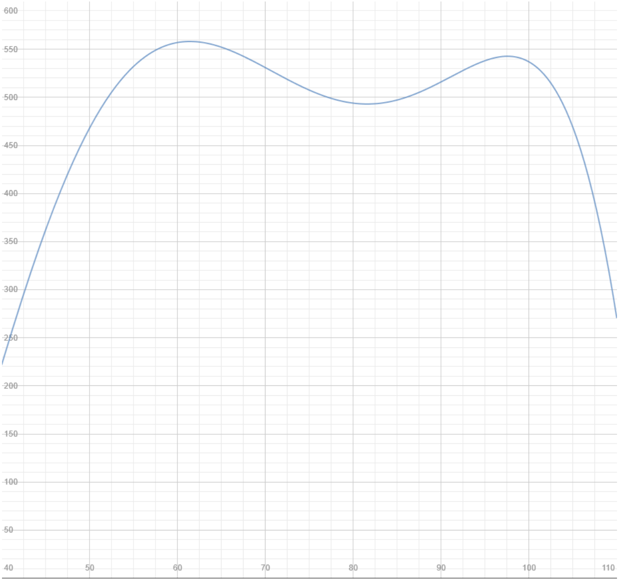
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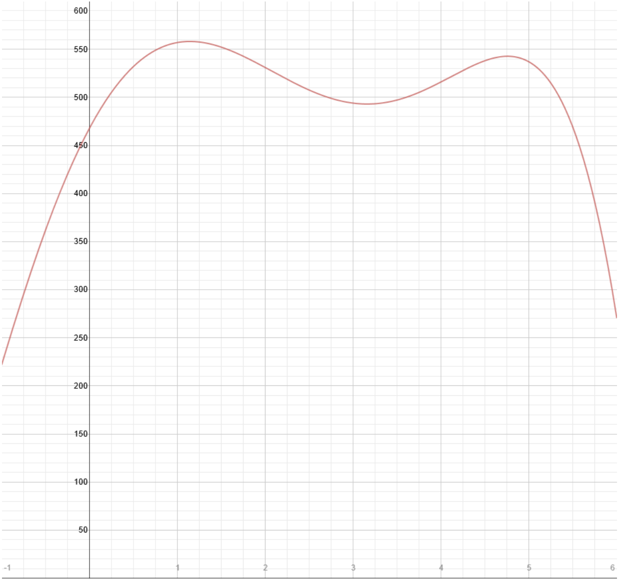

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We realized that the two solutions were just horizontal transformations of one another. But how? The polynomials certainly didn't look like traditional horizontal translations. Eventually, we hit on this idea:

n	x	$g(n) = f(x)$
0	50	468
1	60	557
2	70	531
3	80	494
4	90	516
5	100	537

Looking at this table, we realized that $x = 10n + 50$, and that we could write

$$f(x) = f(10n + 50) = g(n)$$

A quick check on WolframAlpha confirms:

$$\frac{-(10n+50)^5}{125000} + \frac{319(10n+50)^4}{120000} - \frac{2003(10n+50)^3}{6000} + \frac{23219(10n+50)^2}{1200} - \frac{73993(10n+50)}{150} + 4374$$

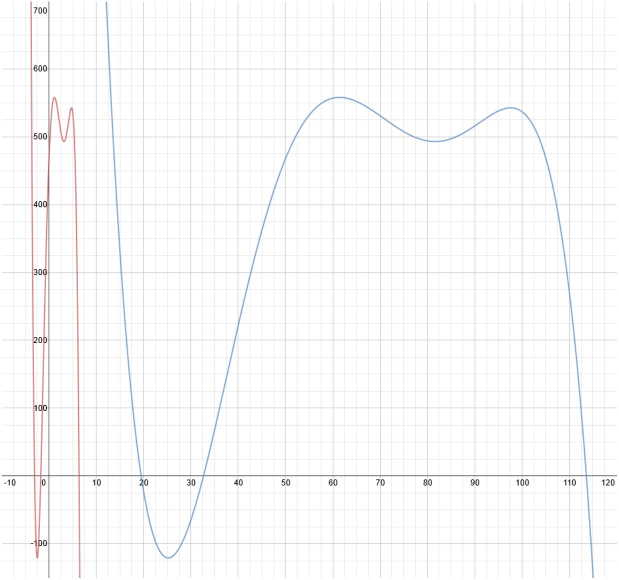
$$= -\frac{4n^5}{5} + \frac{79n^4}{12} - \frac{13n^3}{6} - \frac{1021n^2}{12} + \frac{2557n}{15} + 468$$

Horizontal Transformations are *Backwards!*

In the language of horizontal transformations, we can say

$$f(10x + 50) = g(x)$$

This led to a comment from a student: "I've never understood why horizontal transformations are backwards."



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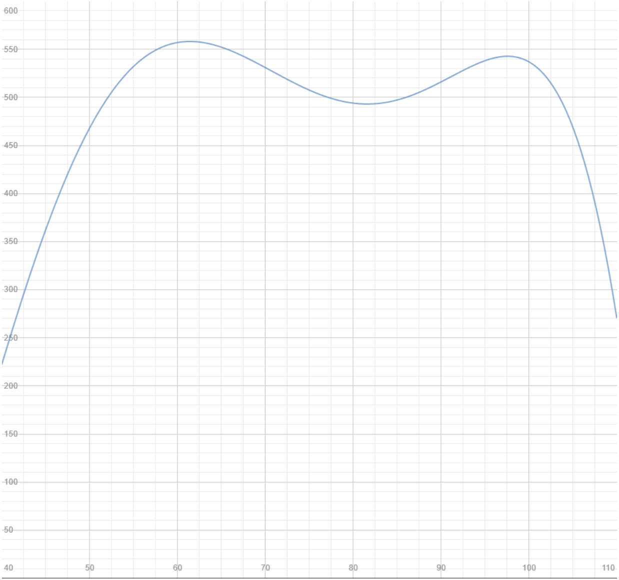
This led to a great class discussion. Some of the ideas students came up with:

1. The function $(10x + 50)$ can be thought of as coming first, like "preprocessing."
2. The x coordinates 0,1,2,3,4,5 take on the values of 50,60,70,80,90,100, after having been *modified* by $(10x + 50)$.
3. Rather than thinking of moving the graph backwards, we can think of shifting the x-axis to the right by 50 units and stretching it by a factor of 10. This makes it look like the graph has moved to the left and shrunk.



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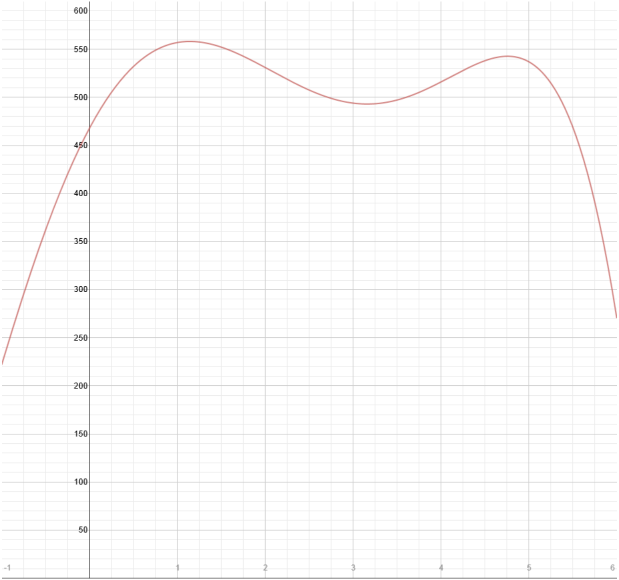

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This class session is one of my all-time favorites, and certainly one of the most memorable.

This class session highlights what I mean by an environment of *group mathematical discovery*:

1. Students are comfortable trying out alternative methods.
2. Students can meaningfully discuss the differences between methods and solutions.
3. Discussion of these differences leads to understanding.

Standards for Mathematical Practice

These ideas are based on several Standards for Mathematical Practice:

SMP 1. Make sense of problems and persevere in solving them.

SMP 2. Reason abstractly and quantitatively.

SMP 3. Construct viable arguments and critique the reasoning of others.

SMP 4. Model with mathematics.

SMP 5. Use appropriate tools strategically.

What can we do to create this environment?

1. *Encourage use of multiple solution methods.*
   One destructive habit many teachers fall into is the habit of insisting on a particular method of solution. Certainly, there are times when we want to teach a specific technique, and we have to insist that students use that technique. However, when it comes to *problem solving,* we should allow students to explore the problem using whatever mathematically useful techniques they have at their disposal.
   (SMP 1,3)

2. *Let students fall down the rabbit hole.*
When I saw my students use the *x*-values of 0 through 5, rather than 50 through 100, I knew that they would not get the back-of-the-book answer. I didn't know where there solution would lead, but (because these were bright students) I could trust them to see the mathematics through to the end. (SMP 1,2,3)

3. *Make your classroom a safe place for exploration.* Your students need to be encouraged to work outside the boundaries of the textbook lessons. They also need to know that they will not be punished for trying something different. They also need to be given the freedom to creatively problem-solve, rather than just churning through exercises.
(SMP 3,4)

4. *Keep technology at the ready.*
   This class session depended on having the ability to quickly simplify complicated polynomials, and graph them. We used WolframAlpha and Desmos on the overhead projector, and students used TI graphing calculators in groups. I have gotten into the habit of loading WolframAlpha and Desmos any time I think I foresee some heavy algebraic manipulation that is not worth doing by hand.
   (SMP 5)

5. *It's okay to ditch the day's lesson plan.*
   The class session I described here was not at all what I had intended on teaching that day. But this class was a rare opportunity, and I didn't want to let it go to waste.

What can we do to recreate this environment?

Ask "what if" questions.

We can't always count on students to come up with novel, thought-provoking questions. Sometimes we need to build these kind of questions into our plans.

A High-School Level Activity

Here's an activity that illustrates the idea of planning surprises. This would be appropriate for students who are just learning about fitting polynomials to data, and already know about horizontal transformations.

The population of Palm Springs was approximately 45,000 in 2003; 48,000 in 2008; and 44,000 in 2013 according to official estimates. We want to find a function for the population as a function of year. Measuring population in thousands of people and year as the number of years after 2000, we have three data points:

$$(3, 45), (8, 48), (13, 44)$$

Here we have the x values $x = 3, 8,$ and 13.

1. Find a polynomial function $f(x)$ that fits these data points.
2. If we measured the years in five-year increments after 2003, our numbers would be easier to work with. 2008 is 1 five-year increment after 2003, and 2013 is 2 five-year increments after 2003. We'll call these new x values x', with $x' = 0, 1,$ and 2. Find a different function, $g(x')$, that fits the data points

$$(0, 45), (1, 48), (2, 44)$$

3. How are these functions similar? How are they different?

4. Write x' as a function of x, and call it $h(x)$. Find $g(h(x))$. What is the result?
5. Write x as a function of x', and call it $j(x')$. Find $f(j(x'))$. What is the result?
6. Graph the functions $f(x), g(x'), g(h(x))$ and $f(j(x'))$. How are these functions related?

Thank you!

Thank you! tcadwall@fullerton.edu