

California Common Core State Standards

Mathematics



COMMON CORE
STATE STANDARDS

CALIFORNIA



*Adopted by the California
State Board of Education
August 2010 and modified
January 2013*

Publishing Information



Senate Bill 1200, Statutes of 2012, called for modification of the California additions to the Common Core State Standards for Mathematics. The California Common Core State Standards: Mathematics (CA CCSSM) were modified January 16, 2013, following the recommendation of State Superintendent of Public Instruction (SSPI) Tom Torlakson. SSPI Torlakson consulted the Mathematics Curriculum Framework and Evaluation Criteria Committee regarding modifications to the CA CCSSM and the organization of model courses in higher mathematics. SSPI Torlakson and the State Board of Education (SBE) convened two public hearings in order for the field to provide input on the recommended modifications. When the CA CCSSM were modified, the members of the SBE were Michael W. Kirst, President; Trish Boyd Williams, Vice President; Sue Burr; Carl A. Cohn; Bruce Holaday; Josephine Kao; Aida Molina; Patricia Ann Rucker; Nicolasa Sandoval; and Ilene Straus.

Senate Bill 1 from the fifth Extraordinary Session (SB X5 1) in 2010 established the California Academic Content Standards Commission (Commission) to evaluate the Common Core State Standards for Mathematics developed by the Common Core State Standards Initiative for rigor and alignment with the California standards. Based on the evaluation, the Commission inserted words, phrases, and select California standards to maintain California's high expectations for students. On July 15, 2010, the Commission recommended that the SBE adopt the CA CCSSM as amended. The members of the Commission were Greg Geeting, Chair; Heather Calahan; Steven Dunlap; Robert Ellis; Eleanor Evans; Bill Evers; Scott Farrand; Mark Freathy; Lori Freiermuth; Bruce Grip; Kathy Harris; Jeanne Jelnick; Deborah Keys; James Lanich; Matt Perry; Pat Sabo; Brian Shay; Alba Sweeney; Hilda Villarreal Writ; Chuck Weis; and Ze'ev Wurman. Support for the Commission was provided by the Sacramento County Office of Education under the direction of Sue Stickle, Deputy Superintendent of Schools.

When the CA CCSSM were adopted by the SBE on August 2, 2010, the members of the SBE were Theodore Mitchell, President; Ruth Bloom, Vice President; Alan Arkatov; James Aschwanden; Benjamin Austin; Yvonne Chan; Gregory Jones; David Lopez; and Johnathan Williams. Jack O'Connell, former State Superintendent of Public Instruction, is also recognized for his leadership during the adoption of the standards in August 2010.

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Special recognition is awarded to Joy Kessel, Analyst, Common Core Systems Implementation Office, for her contribution to the original organization and format design.

Ordering Information

Copies of the *California Common Core State Standards: Mathematics* are available for purchase from the California Department of Education. For prices and ordering information, please visit the Department Web site at <http://www.cde.ca.gov/re/pn/rc> or call the CDE Press Sales Office at 1-800-995-4099.

Notice

The guidance in the *California Common Core State Standards: Mathematics* is not binding on local educational agencies or other entities. Except for the statutes, regulations, and court decisions that are referenced herein, the document is exemplary, and compliance with it is not mandatory. (See *California Education Code* Section 33308.5).



A Message from the State Board of Education and the State Superintendent of Public Instruction

The California Common Core State Standards: Mathematics (CA CCSSM) reflect the importance of focus, coherence, and rigor as the guiding principles for mathematics instruction and learning. California's implementation of the CA CCSSM demonstrates a commitment to providing a world-class education for all students that supports college and career readiness and the knowledge and skills necessary to fully participate in the twenty-first-century global economy.

The CA CCSSM build on California's standards-based educational system in which curriculum, instruction, professional learning, assessment, and accountability are aligned to support student attainment of the standards. The CA CCSSM incorporate current research and input from education stakeholders—including other state departments of education, scholars, professional organizations, teachers and other educators, parents, and students. California additions to the standards (identified in boldface text and followed by the abbreviation "CA") were incorporated in an effort to retain the consistency and precision of our past standards. The CA CCSSM are internationally benchmarked, research-based, and unequivocally rigorous.

The standards call for learning mathematical content in the context of real-world situations, using mathematics to solve problems, and developing "habits of mind" that foster mastery of mathematics content as well as mathematical understanding. The standards for kindergarten through grade 8 prepare students for higher mathematics. The standards for higher mathematics reflect the knowledge and skills that are necessary to prepare students for college and careers and productive citizenship.

Implementation of the CA CCSSM will take time and effort, but it also provides a new and exciting opportunity to ensure that California's students are held to the same high expectations in mathematics as their national and global peers. Although California educators have implemented standards before, the CA CCSSM require not only rigorous curriculum and instruction but also conceptual understanding, procedural skill and fluency, and the ability to apply mathematics. In short, the standards call for meeting the challenges of the twenty-first century through innovation.

Handwritten signature of Michael W. Kirst in black ink.

MICHAEL W. KIRST, President
California State Board of Education

Handwritten signature of Tom Torlakson in black ink.

TOM TORLAKSON
State Superintendent of Public Instruction

Introduction



All students need a high-quality mathematics program designed to prepare them to graduate from high school ready for college and careers. In support of this goal, California adopted the California Common Core State Standards: Mathematics (CA CCSSM) in August 2010, replacing the 1997 statewide mathematics academic standards. As part of the modification of the CA CCSSM in January 2013, the California State Board of Education also approved higher mathematics standards organized into model courses.

The CA CCSSM are designed to be robust, linked within and across grades, and relevant to the real world, reflecting the knowledge and skills that young people will need for success in college and careers. With California's students fully prepared for the future, our students will be positioned to compete successfully in the global economy.

The development of the standards began as a voluntary, state-led effort coordinated by the Council of Chief State School Officers (CCSSO) and the National Governors Association (NGA) Center for Best Practices. Both organizations were committed to developing a set of standards that would help prepare students for success in career and college. The CA CCSSM are based on evidence of the skills and knowledge needed for college and career readiness and an expectation that students be able to know and do mathematics by solving a range of problems and engaging in key mathematical practices.

The development of the standards was informed by international benchmarking and began with research on what is known about how students' mathematical knowledge, skills, and understanding develop over time. The progression from kindergarten standards to standards for higher mathematics exemplifies the three principles of focus, coherence, and rigor that are the basis of the CCSSM.

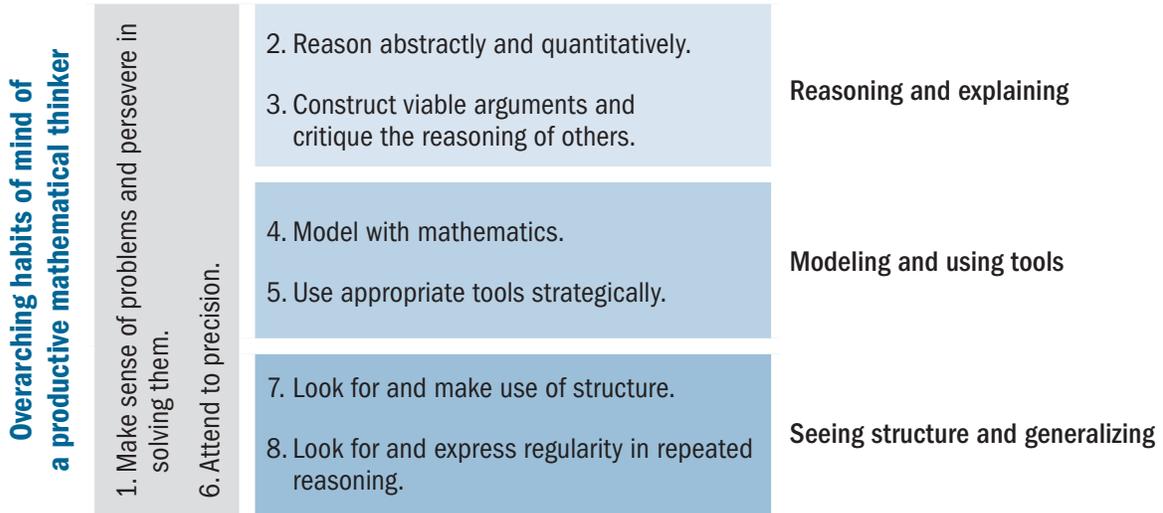
The first principle, *focus*, means that instruction should focus deeply on only those concepts that are emphasized in the standards so that students can gain strong foundational conceptual understanding, a high degree of procedural skill and fluency, and the ability to apply the mathematics they know to solve problems inside and outside the mathematics classroom. *Coherence* arises from mathematical connections. Some of the connections in the standards knit topics together at a single grade level. Most connections are vertical, as the standards support a progression of increasing knowledge, skill, and sophistication across the grades. Finally, *rigor* requires that conceptual understanding, procedural skill and fluency, and application be approached with equal intensity.

Two Types of Standards

The CA CCSSM include two types of standards: Eight Mathematical Practice Standards (identical for each grade level) and Mathematical Content Standards (different at each grade level). Together these standards address both "habits of mind" that students should develop to foster mathematical understanding and expertise and skills and knowledge—what students need to know and be able to do. The mathematical content standards were built on progressions of topics across grade levels, informed by both research on children's cognitive development and by the logical structure of mathematics.

The Standards for Mathematical Practice (MP) are the same at each grade level, with the exception of an additional practice standard included in the CA CCSSM for higher mathematics only: **MP3.1: Students build proofs by induction and proofs by contradiction.** CA This standard may be seen as an extension of Mathematical Practice 3, in which students construct viable arguments and critique the reasoning of others. Ideally, several MP standards will be evident in each lesson as they interact and overlap with each other. The MP standards are not a checklist; they are the basis of mathematics instruction and learning. Structuring the MP standards can help educators recognize opportunities for students to engage with mathematics in grade-appropriate ways. The eight MP standards may be grouped into four categories as illustrated in the following chart.

Structuring the Standards for Mathematical Practice¹



The CA CCSSM call for mathematical practices and mathematical content to be connected as students engage in mathematical tasks. These connections are essential to support the development of students' broader mathematical understanding—students who lack understanding of a topic may rely too heavily on procedures. The MP standards must be taught as carefully and practiced as intentionally as the Standards for Mathematical Content. Neither should be isolated from the other; effective mathematics instruction occurs when the two halves of the CA CCSSM come together as a powerful whole.

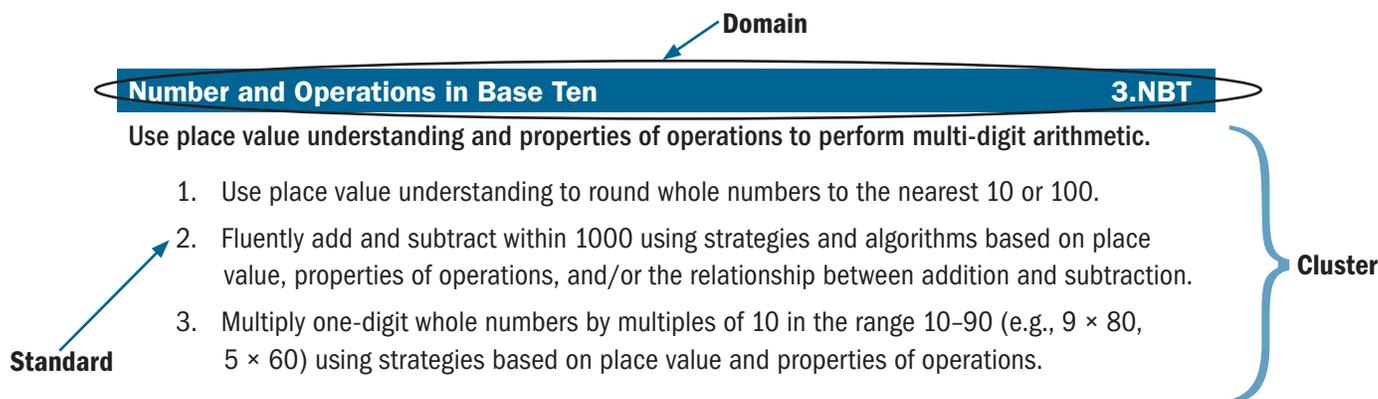
How to Read the Standards

Kindergarten–Grade 8

In kindergarten through grade 8, the CA CCSSM are organized by grade level and then by domains (clusters of standards that address “big ideas” and support connections of topics across the grades), clusters (groups of related standards inside domains), and finally by the standards (what students should understand and be able to do). The standards do not dictate curriculum or pedagogy. For example, just because Topic A appears before Topic B in the standards for a given grade does not mean that Topic A must be taught before Topic B.

The code for each standard begins with the grade level, followed by the domain code and the number of the standard. For example, “3.NBT 2” would be the second standard in the domain of Number and Operations in Base Ten of the standards for grade 3.

1. Bill McCallum. 2011. *Structuring the Mathematical Practices*. <http://commoncoretools.me/wp-content/uploads/2011/03/practices.pdf> (accessed April 1, 2013).



Higher Mathematics

In California, the CA CCSSM for higher mathematics are organized into both model courses and conceptual categories. The higher mathematics courses adopted by the State Board of Education in January 2013 are based on the guidance provided in Appendix A published by the Common Core State Standards Initiative.² The model courses for higher mathematics are organized into two pathways: traditional and integrated. The traditional pathway consists of the higher mathematics standards organized along more traditional lines into Algebra I, Geometry, and Algebra II courses. The integrated pathway consists of the courses Mathematics I, II, and III. The integrated pathway presents higher mathematics as a connected subject, in that each course contains standards from all six of the conceptual categories. In addition, two advanced higher mathematics courses were retained from the 1997 mathematics standards: Advanced Placement Probability and Statistics and Calculus.

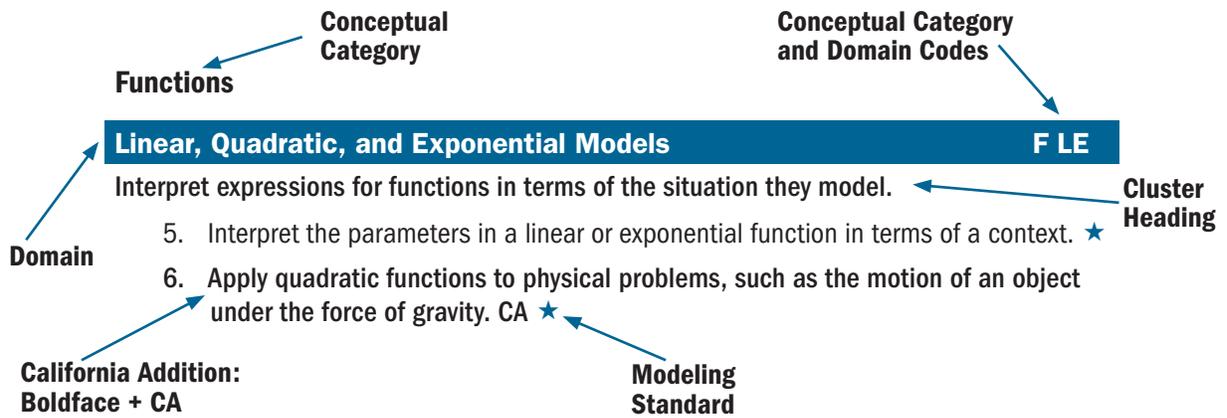
The standards for higher mathematics are also organized into conceptual categories:

- Number and Quantity
- Algebra
- Functions
- Modeling
- Geometry
- Statistics and Probability

The conceptual categories portray a coherent view of higher mathematics based on the realization that students' work on a broad topic, such as functions, crosses a number of traditional course boundaries. As local school districts develop a full range of courses and curriculum in higher mathematics, the organization of standards by conceptual categories offers a starting point for discussing course content.

The code for each higher mathematics standard begins with the identifier for the conceptual category code (N, A, F, G, S), followed by the domain code and the number of the standard. For example, "F-LE.5" would be the fifth standard in the domain of Linear, Quadratic, and Exponential Models in the conceptual category of Functions.

² Appendix A provides guidance to the field on developing higher mathematics courses. This appendix is available on the Common Core State Standards Initiative Web site at <http://www.corestandards.org/Math>.



The star symbol (★) following the standard indicates that it is also a Modeling standard. Modeling is best interpreted not as a collection of isolated topics but in relation to other standards. Making mathematical models is an MP standard, and modeling standards appear throughout the higher mathematics standards indicated by a ★ symbol. Additional mathematics that students should learn in order to take advanced courses such as calculus, advanced statistics, or discrete mathematics is indicated by a plus symbol (+). Standards with a (+) symbol may appear in courses intended for all students.

Standards for Mathematical Practice



The Standards for Mathematical Practice describe varieties of expertise that mathematics educators at all levels should seek to develop in their students. These practices rest on important “processes and proficiencies” with longstanding importance in mathematics education. The first of these are the NCTM process standards of problem solving, reasoning and proof, communication, representation, and connections. The second are the strands of mathematical proficiency specified in the National Research Council’s report *Adding It Up*: adaptive reasoning, strategic competence, conceptual understanding (comprehension of mathematical concepts, operations and relations), procedural fluency (skill in carrying out procedures flexibly, accurately, efficiently and appropriately), and productive disposition (habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one’s own efficacy).

1) Make sense of problems and persevere in solving them.

Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, “Does this make sense?” They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

2) Reason abstractly and quantitatively.

Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to *decontextualize*—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to *contextualize*, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

3) Construct viable arguments and critique the reasoning of others.

Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine domains to which an

argument applies. Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments. **Students build proofs by induction and proofs by contradiction. CA 3.1** (for higher mathematics only).

4) Model with mathematics.

Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

5) Use appropriate tools strategically.

Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. Proficient students are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, mathematically proficient high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. Mathematically proficient students at various grade levels are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.

6) Attend to precision.

Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context. In the elementary grades, students give carefully formulated explanations to each other. By the time they reach high school they have learned to examine claims and make explicit use of definitions.

7) Look for and make use of structure.

Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7×8 equals the well-remembered $7 \times 5 + 7 \times 3$, in preparation for learning about the distributive property. In the expression $x^2 + 9x + 14$, older students can see the 14 as 2×7 and the 9 as $2 + 7$.

They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 - 3(x - y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers x and y .

8) Look for and express regularity in repeated reasoning.

Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1, 2) with slope 3, middle school students might abstract the equation $(y - 2)/(x - 1) = 3$. Noticing the regularity in the way terms cancel when expanding $(x - 1)(x + 1)$, $(x - 1)(x^2 + x + 1)$, and $(x - 1)(x^3 + x^2 + x + 1)$ might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

Connecting the Standards for Mathematical Practice to the Standards for Mathematical Content

The Standards for Mathematical Practice describe ways in which developing student practitioners of the discipline of mathematics increasingly ought to engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle and high school years. Designers of curricula, assessments, and professional development should all attend to the need to connect the mathematical practices to mathematical content in mathematics instruction.

The Standards for Mathematical Content are a balanced combination of procedure and understanding. Expectations that begin with the word “understand” are often especially good opportunities to connect the practices to the content. Students who lack understanding of a topic may rely on procedures too heavily. Without a flexible base from which to work, they may be less likely to consider analogous problems, represent problems coherently, justify conclusions, apply the mathematics to practical situations, use technology mindfully to work with the mathematics, explain the mathematics accurately to other students, step back for an overview, or deviate from a known procedure to find a shortcut. In short, a lack of understanding effectively prevents a student from engaging in the mathematical practices.

In this respect, those content standards which set an expectation of understanding are potential “points of intersection” between the Standards for Mathematical Content and the Standards for Mathematical Practice. These points of intersection are intended to be weighted toward central and generative concepts in the school mathematics curriculum that most merit the time, resources, innovative energies, and focus necessary to qualitatively improve the curriculum, instruction, assessment, professional development, and student achievement in mathematics.

Kindergarten



In kindergarten, instructional time should focus on two critical areas: (1) representing, relating, and operating on whole numbers, initially with sets of objects; and (2) describing shapes and space. More learning time in kindergarten should be devoted to number than to other topics.

- (1) Students use numbers, including written numerals, to represent quantities and to solve quantitative problems, such as counting objects in a set; counting out a given number of objects; comparing sets or numerals; and modeling simple joining and separating situations with sets of objects, or eventually with equations such as $5 + 2 = 7$ and $7 - 2 = 5$. (Kindergarten students should see addition and subtraction equations, and student writing of equations in kindergarten is encouraged, but it is not required.) Students choose, combine, and apply effective strategies for answering quantitative questions, including quickly recognizing the cardinalities of small sets of objects, counting and producing sets of given sizes, counting the number of objects in combined sets, or counting the number of objects that remain in a set after some are taken away.
- (2) Students describe their physical world using geometric ideas (e.g., shape, orientation, spatial relations) and vocabulary. They identify, name, and describe basic two-dimensional shapes, such as squares, triangles, circles, rectangles, and hexagons, presented in a variety of ways (e.g., with different sizes and orientations), as well as three-dimensional shapes such as cubes, cones, cylinders, and spheres. They use basic shapes and spatial reasoning to model objects in their environment and to construct more complex shapes.

Kindergarten Overview

Counting and Cardinality

- Know number names and the count sequence.
- Count to tell the number of objects.
- Compare numbers.

Operations and Algebraic Thinking

- Understand addition as putting together and adding to, and understand subtraction as taking apart and taking from.

Number and Operations in Base Ten

- Work with numbers 11–19 to gain foundations for place value.

Measurement and Data

- Describe and compare measurable attributes.
- Classify objects and count the number of objects in categories.

Geometry

- Identify and describe shapes.
- Analyze, compare, create, and compose shapes.

Mathematical Practices

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Counting and Cardinality**K.CC****Know number names and the count sequence.**

1. Count to 100 by ones and by tens.
2. Count forward beginning from a given number within the known sequence (instead of having to begin at 1).
3. Write numbers from 0 to 20. Represent a number of objects with a written numeral 0–20 (with 0 representing a count of no objects).

Count to tell the number of objects.

4. Understand the relationship between numbers and quantities; connect counting to cardinality.
 - a. When counting objects, say the number names in the standard order, pairing each object with one and only one number name and each number name with one and only one object.
 - b. Understand that the last number name said tells the number of objects counted. The number of objects is the same regardless of their arrangement or the order in which they were counted.
 - c. Understand that each successive number name refers to a quantity that is one larger.
5. Count to answer “how many?” questions about as many as 20 things arranged in a line, a rectangular array, or a circle, or as many as 10 things in a scattered configuration; given a number from 1–20, count out that many objects.

Compare numbers.

6. Identify whether the number of objects in one group is greater than, less than, or equal to the number of objects in another group, e.g., by using matching and counting strategies.¹
7. Compare two numbers between 1 and 10 presented as written numerals.

Operations and Algebraic Thinking**K.OA****Understand addition as putting together and adding to, and understand subtraction as taking apart and taking from.**

1. Represent addition and subtraction with objects, fingers, mental images, drawings,² sounds (e.g., claps), acting out situations, verbal explanations, expressions, or equations.
2. Solve addition and subtraction word problems, and add and subtract within 10, e.g., by using objects or drawings to represent the problem.
3. Decompose numbers less than or equal to 10 into pairs in more than one way, e.g., by using objects or drawings, and record each decomposition by a drawing or equation (e.g., $5 = 2 + 3$ and $5 = 4 + 1$).
4. For any number from 1 to 9, find the number that makes 10 when added to the given number, e.g., by using objects or drawings, and record the answer with a drawing or equation.
5. Fluently add and subtract within 5.

1. Includes groups with up to ten objects.

2. Drawings need not show details, but should show the mathematics in the problem. (This applies wherever drawings are mentioned in the Standards.)

Number and Operations in Base Ten

K.NBT

Work with numbers 11–19 to gain foundations for place value.

1. Compose and decompose numbers from 11 to 19 into ten ones and some further ones, e.g., by using objects or drawings, and record each composition or decomposition by a drawing or equation (e.g., $18 = 10 + 8$); understand that these numbers are composed of ten ones and one, two, three, four, five, six, seven, eight, or nine ones.

Measurement and Data

K.MD

Describe and compare measurable attributes.

1. Describe measurable attributes of objects, such as length or weight. Describe several measurable attributes of a single object.
2. Directly compare two objects with a measurable attribute in common, to see which object has “more of”/“less of” the attribute, and describe the difference. *For example, directly compare the heights of two children and describe one child as taller/shorter.*

Classify objects and count the number of objects in each category.

3. Classify objects into given categories; count the numbers of objects in each category and sort the categories by count.³

Geometry

K.G

Identify and describe shapes (squares, circles, triangles, rectangles, hexagons, cubes, cones, cylinders, and spheres).

1. Describe objects in the environment using names of shapes, and describe the relative positions of these objects using terms such as *above*, *below*, *beside*, *in front of*, *behind*, and *next to*.
2. Correctly name shapes regardless of their orientations or overall size.
3. Identify shapes as two-dimensional (lying in a plane, “flat”) or three-dimensional (“solid”).

Analyze, compare, create, and compose shapes.

4. Analyze and compare two- and three-dimensional shapes, in different sizes and orientations, using informal language to describe their similarities, differences, parts (e.g., number of sides and vertices/“corners”) and other attributes (e.g., having sides of equal length).
5. Model shapes in the world by building shapes from components (e.g., sticks and clay balls) and drawing shapes.
6. Compose simple shapes to form larger shapes. *For example, “Can you join these two triangles with full sides touching to make a rectangle?”*

3. Limit category counts to be less than or equal to 10.

Grade 1



In grade 1, instructional time should focus on four critical areas: (1) developing understanding of addition, subtraction, and strategies for addition and subtraction within 20; (2) developing understanding of whole number relationships and place value, including grouping in tens and ones; (3) developing understanding of linear measurement and measuring lengths as iterating length units; and (4) reasoning about attributes of, and composing and decomposing geometric shapes.

- (1) Students develop strategies for adding and subtracting whole numbers based on their prior work with small numbers. They use a variety of models, including discrete objects and length-based models (e.g., cubes connected to form lengths), to model add-to, take-from, put-together, take-apart, and compare situations to develop meaning for the operations of addition and subtraction, and to develop strategies to solve arithmetic problems with these operations. Students understand connections between counting and addition and subtraction (e.g., adding two is the same as counting on two). They use properties of addition to add whole numbers and to create and use increasingly sophisticated strategies based on these properties (e.g., “making tens”) to solve addition and subtraction problems within 20. By comparing a variety of solution strategies, children build their understanding of the relationship between addition and subtraction.
- (2) Students develop, discuss, and use efficient, accurate, and generalizable methods to add within 100 and subtract multiples of 10. They compare whole numbers (at least to 100) to develop understanding of and solve problems involving their relative sizes. They think of whole numbers between 10 and 100 in terms of tens and ones (especially recognizing the numbers 11 to 19 as composed of a ten and some ones). Through activities that build number sense, they understand the order of the counting numbers and their relative magnitudes.
- (3) Students develop an understanding of the meaning and processes of measurement, including underlying concepts such as iterating (the mental activity of building up the length of an object with equal-sized units) and the transitivity principle for indirect measurement.¹
- (4) Students compose and decompose plane or solid figures (e.g., put two triangles together to make a quadrilateral) and build understanding of part-whole relationships as well as the properties of the original and composite shapes. As they combine shapes, they recognize them from different perspectives and orientations, describe their geometric attributes, and determine how they are alike and different, to develop the background for measurement and for initial understandings of properties such as congruence and symmetry.

1. Students should apply the principle of transitivity of measurement to make indirect comparisons, but they need not use this technical term.

Grade 1 Overview

Operations and Algebraic Thinking

- Represent and solve problems involving addition and subtraction.
- Understand and apply properties of operations and the relationship between addition and subtraction.
- Add and subtract within 20.
- Work with addition and subtraction equations.

Number and Operations in Base Ten

- Extend the counting sequence.
- Understand place value.
- Use place value understanding and properties of operations to add and subtract.

Measurement and Data

- Measure lengths indirectly and by iterating length units.
- Tell and write time.
- Represent and interpret data.

Geometry

- Reason with shapes and their attributes.

Mathematical Practices

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Operations and Algebraic Thinking

1.OA

Represent and solve problems involving addition and subtraction.

1. Use addition and subtraction within 20 to solve word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions, e.g., by using objects, drawings, and equations with a symbol for the unknown number to represent the problem.²
2. Solve word problems that call for addition of three whole numbers whose sum is less than or equal to 20, e.g., by using objects, drawings, and equations with a symbol for the unknown number to represent the problem.

Understand and apply properties of operations and the relationship between addition and subtraction.

3. Apply properties of operations as strategies to add and subtract.³ *Examples: If $8 + 3 = 11$ is known, then $3 + 8 = 11$ is also known. (Commutative property of addition.) To add $2 + 6 + 4$, the second two numbers can be added to make a ten, so $2 + 6 + 4 = 2 + 10 = 12$. (Associative property of addition.)*
4. Understand subtraction as an unknown-addend problem. *For example, subtract $10 - 8$ by finding the number that makes 10 when added to 8.*

Add and subtract within 20.

5. Relate counting to addition and subtraction (e.g., by counting on 2 to add 2).
6. Add and subtract within 20, demonstrating fluency for addition and subtraction within 10. Use strategies such as counting on; making ten (e.g., $8 + 6 = 8 + 2 + 4 = 10 + 4 = 14$); decomposing a number leading to a ten (e.g., $13 - 4 = 13 - 3 - 1 = 10 - 1 = 9$); using the relationship between addition and subtraction (e.g., knowing that $8 + 4 = 12$, one knows $12 - 8 = 4$); and creating equivalent but easier or known sums (e.g., adding $6 + 7$ by creating the known equivalent $6 + 6 + 1 = 12 + 1 = 13$).

Work with addition and subtraction equations.

7. Understand the meaning of the equal sign, and determine if equations involving addition and subtraction are true or false. *For example, which of the following equations are true and which are false? $6 = 6$, $7 = 8 - 1$, $5 + 2 = 2 + 5$, $4 + 1 = 5 + 2$.*
8. Determine the unknown whole number in an addition or subtraction equation relating three whole numbers. *For example, determine the unknown number that makes the equation true in each of the equations $8 + \square = 11$, $5 = \square - 3$, $6 + 6 = \square$.*

Number and Operations in Base Ten

1.NBT

Extend the counting sequence.

1. Count to 120, starting at any number less than 120. In this range, read and write numerals and represent a number of objects with a written numeral.

Understand place value.

2. Understand that the two digits of a two-digit number represent amounts of tens and ones. Understand the following as special cases:
 - a. 10 can be thought of as a bundle of ten ones—called a “ten.”
 - b. The numbers from 11 to 19 are composed of a ten and one, two, three, four, five, six, seven, eight, or nine ones.
 - c. The numbers 10, 20, 30, 40, 50, 60, 70, 80, 90 refer to one, two, three, four, five, six, seven, eight, or nine tens (and 0 ones).

2. See Glossary, Table 1.

3. Students need not use formal terms for these properties.

3. Compare two two-digit numbers based on meanings of the tens and ones digits, recording the results of comparisons with the symbols $>$, $=$, and $<$.

Use place value understanding and properties of operations to add and subtract.

4. Add within 100, including adding a two-digit number and a one-digit number, and adding a two-digit number and a multiple of 10, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used. Understand that in adding two-digit numbers, one adds tens and tens, ones and ones; and sometimes it is necessary to compose a ten.
5. Given a two-digit number, mentally find 10 more or 10 less than the number, without having to count; explain the reasoning used.
6. Subtract multiples of 10 in the range 10–90 from multiples of 10 in the range 10–90 (positive or zero differences), using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used.

Measurement and Data

1.MD

Measure lengths indirectly and by iterating length units.

1. Order three objects by length; compare the lengths of two objects indirectly by using a third object.
2. Express the length of an object as a whole number of length units, by laying multiple copies of a shorter object (the length unit) end to end; understand that the length measurement of an object is the number of same-size length units that span it with no gaps or overlaps. *Limit to contexts where the object being measured is spanned by a whole number of length units with no gaps or overlaps.*

Tell and write time.

3. Tell and write time in hours and half-hours using analog and digital clocks.

Represent and interpret data.

4. Organize, represent, and interpret data with up to three categories; ask and answer questions about the total number of data points, how many in each category, and how many more or less are in one category than in another.

Geometry

1.G

Reason with shapes and their attributes.

1. Distinguish between defining attributes (e.g., triangles are closed and three-sided) versus non-defining attributes (e.g., color, orientation, overall size); build and draw shapes to possess defining attributes.
2. Compose two-dimensional shapes (rectangles, squares, trapezoids, triangles, half-circles, and quarter-circles) or three-dimensional shapes (cubes, right rectangular prisms, right circular cones, and right circular cylinders) to create a composite shape, and compose new shapes from the composite shape.⁴
3. Partition circles and rectangles into two and four equal shares, describe the shares using the words *halves*, *fourths*, and *quarters*, and use the phrases *half of*, *fourth of*, and *quarter of*. Describe the whole as two of, or four of the shares. Understand for these examples that decomposing into more equal shares creates smaller shares.

4. Students do not need to learn formal names such as “right rectangular prism.”

Grade 2



In grade 2, instructional time should focus on four critical areas: (1) extending understanding of base-ten notation; (2) building fluency with addition and subtraction; (3) using standard units of measure; and (4) describing and analyzing shapes.

- (1) Students extend their understanding of the base-ten system. This includes ideas of counting in fives, tens, and multiples of hundreds, tens, and ones, as well as number relationships involving these units, including comparing. Students understand multi-digit numbers (up to 1000) written in base-ten notation, recognizing that the digits in each place represent amounts of thousands, hundreds, tens, or ones (e.g., 853 is 8 hundreds + 5 tens + 3 ones).
- (2) Students use their understanding of addition to develop fluency with addition and subtraction within 100. They solve problems within 1000 by applying their understanding of models for addition and subtraction, and they develop, discuss, and use efficient, accurate, and generalizable methods to compute sums and differences of whole numbers in base-ten notation, using their understanding of place value and the properties of operations. They select and accurately apply methods that are appropriate for the context and the numbers involved to mentally calculate sums and differences for numbers with only tens or only hundreds.
- (3) Students recognize the need for standard units of measure (centimeter and inch) and they use rulers and other measurement tools with the understanding that linear measure involves an iteration of units. They recognize that the smaller the unit, the more iterations they need to cover a given length.
- (4) Students describe and analyze shapes by examining their sides and angles. Students investigate, describe, and reason about decomposing and combining shapes to make other shapes. Through building, drawing, and analyzing two- and three-dimensional shapes, students develop a foundation for understanding area, volume, congruence, similarity, and symmetry in later grades.

Grade 2 Overview

Operations and Algebraic Thinking

- Represent and solve problems involving addition and subtraction.
- Add and subtract within 20.
- Work with equal groups of objects to gain foundations for multiplication.

Number and Operations in Base Ten

- Understand place value.
- Use place value understanding and properties of operations to add and subtract.

Measurement and Data

- Measure and estimate lengths in standard units.
- Relate addition and subtraction to length.
- Work with time and money.
- Represent and interpret data.

Geometry

- Reason with shapes and their attributes.

Mathematical Practices

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Operations and Algebraic Thinking

2.OA

Represent and solve problems involving addition and subtraction.

1. Use addition and subtraction within 100 to solve one- and two-step word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem.¹

Add and subtract within 20.

2. Fluently add and subtract within 20 using mental strategies.² By end of Grade 2, know from memory all sums of two one-digit numbers.

Work with equal groups of objects to gain foundations for multiplication.

3. Determine whether a group of objects (up to 20) has an odd or even number of members, e.g., by pairing objects or counting them by 2s; write an equation to express an even number as a sum of two equal addends.
4. Use addition to find the total number of objects arranged in rectangular arrays with up to 5 rows and up to 5 columns; write an equation to express the total as a sum of equal addends.

Number and Operations in Base Ten

2.NBT

Understand place value.

1. Understand that the three digits of a three-digit number represent amounts of hundreds, tens, and ones; e.g., 706 equals 7 hundreds, 0 tens, and 6 ones. Understand the following as special cases:
 - a. 100 can be thought of as a bundle of ten tens—called a “hundred.”
 - b. The numbers 100, 200, 300, 400, 500, 600, 700, 800, 900 refer to one, two, three, four, five, six, seven, eight, or nine hundreds (and 0 tens and 0 ones).
2. Count within 1000; skip-count by 2s, 5s, 10s, and 100s. CA
3. Read and write numbers to 1000 using base-ten numerals, number names, and expanded form.
4. Compare two three-digit numbers based on meanings of the hundreds, tens, and ones digits, using $>$, $=$, and $<$ symbols to record the results of comparisons.

Use place value understanding and properties of operations to add and subtract.

5. Fluently add and subtract within 100 using strategies based on place value, properties of operations, and/or the relationship between addition and subtraction.
6. Add up to four two-digit numbers using strategies based on place value and properties of operations.
7. Add and subtract within 1000, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method. Understand that in adding or subtracting three-digit numbers, one adds or subtracts hundreds and hundreds, tens and tens, ones and ones; and sometimes it is necessary to compose or decompose tens or hundreds.

1. See Glossary, Table 1.

2. See standard 1.OA.6 for a list of mental strategies.

7.1 Use estimation strategies to make reasonable estimates in problem solving. CA

8. Mentally add 10 or 100 to a given number 100–900, and mentally subtract 10 or 100 from a given number 100–900.
9. Explain why addition and subtraction strategies work, using place value and the properties of operations.³

Measurement and Data**2.MD****Measure and estimate lengths in standard units.**

1. Measure the length of an object by selecting and using appropriate tools such as rulers, yardsticks, meter sticks, and measuring tapes.
2. Measure the length of an object twice, using length units of different lengths for the two measurements; describe how the two measurements relate to the size of the unit chosen.
3. Estimate lengths using units of inches, feet, centimeters, and meters.
4. Measure to determine how much longer one object is than another, expressing the length difference in terms of a standard length unit.

Relate addition and subtraction to length.

5. Use addition and subtraction within 100 to solve word problems involving lengths that are given in the same units, e.g., by using drawings (such as drawings of rulers) and equations with a symbol for the unknown number to represent the problem.
6. Represent whole numbers as lengths from 0 on a number line diagram with equally spaced points corresponding to the numbers 0, 1, 2, . . . , and represent whole-number sums and differences within 100 on a number line diagram.

Work with time and money.

7. Tell and write time from analog and digital clocks to the nearest five minutes, using a.m. and p.m. **Know relationships of time (e.g., minutes in an hour, days in a month, weeks in a year). CA**
8. Solve word problems involving dollar bills, quarters, dimes, nickels, and pennies, using \$ and ¢ symbols appropriately.
Example: If you have 2 dimes and 3 pennies, how many cents do you have?

Represent and interpret data.

9. Generate measurement data by measuring lengths of several objects to the nearest whole unit, or by making repeated measurements of the same object. Show the measurements by making a line plot, where the horizontal scale is marked off in whole-number units.
10. Draw a picture graph and a bar graph (with single-unit scale) to represent a data set with up to four categories. Solve simple put-together, take-apart, and compare problems⁴ using information presented in a bar graph.

3. Explanations may be supported by drawings or objects.

4. See Glossary, Table 1.

Reason with shapes and their attributes.

1. Recognize and draw shapes having specified attributes, such as a given number of angles or a given number of equal faces.⁵ Identify triangles, quadrilaterals, pentagons, hexagons, and cubes.
2. Partition a rectangle into rows and columns of same-size squares and count to find the total number of them.
3. Partition circles and rectangles into two, three, or four equal shares, describe the shares using the words *halves*, *thirds*, *half of*, *a third of*, etc., and describe the whole as two halves, three thirds, four fourths. Recognize that equal shares of identical wholes need not have the same shape.

5. Sizes are compared directly or visually, not compared by measuring.

Glossary



Addition and subtraction within 5, 10, 20, 100, or 1000. Addition or subtraction of two whole numbers with whole number answers, and with sum or minuend in the range 0-5, 0-10, 0-20, or 0-100, respectively. Example: $8 + 2 = 10$ is an addition within 10, $14 - 5 = 9$ is a subtraction within 20, and $55 - 18 = 37$ is a subtraction within 100.

Additive inverses. Two numbers whose sum is 0 are additive inverses of one another. Example: $3/4$ and $-3/4$ are additive inverses of one another because $3/4 + (-3/4) = (-3/4) + 3/4 = 0$.

Associative property of addition. See Table 3 in this Glossary.

Associative property of multiplication. See Table 3 in this Glossary.

Bivariate data. Pairs of linked numerical observations. Example: a list of heights and weights for each player on a football team.

Box plot. A method of visually displaying a distribution of data values by using the median, quartiles, and extremes of the data set. A box shows the middle 50% of the data.

Commutative property. See Table 3 in this Glossary.

Complex fraction. A fraction A/B where A and/or B are fractions (B nonzero).

Computation algorithm. A set of predefined steps applicable to a class of problems that gives the correct result in every case when the steps are carried out correctly. *See also:* computation strategy.

Computation strategy. Purposeful manipulations that may be chosen for specific problems, may not have a fixed order, and may be aimed at converting one problem into another. *See also:* computation algorithm.

Congruent. Two plane or solid figures are congruent if one can be obtained from the other by rigid motion (a sequence of rotations, reflections, and translations).

Counting on. A strategy for finding the number of objects in a group without having to count every member of the group. For example, if a stack of books is known to have 8 books and 3 more books are added to the top, it is not necessary to count the stack all over again. One can find the total by *counting on*—pointing to the top book and saying “eight,” following this with “nine, ten, eleven. There are eleven books now.”

Dot plot. *See:* line plot.

Dilation. A transformation that moves each point along the ray through the point emanating from a fixed center, and multiplies distances from the center by a common scale factor.

Expanded form. A multi-digit number is expressed in expanded form when it is written as a sum of single-digit multiples of powers of ten. For example, $643 = 600 + 40 + 3$.

Expected value. For a random variable, the weighted average of its possible values, with weights given by their respective probabilities.

First quartile. For a data set with median M , the first quartile is the median of the data values less than M . Example: For the data set {1, 3, 6, 7, 10, 12, 14, 15, 22, 120}, the first quartile is 6.¹ See *also*: median, third quartile, interquartile range.

Fraction. A number expressible in the form a/b where a is a whole number and b is a positive whole number. (The word *fraction* in these standards always refers to a non-negative number.) See *also*: rational number.

Identity property of 0. See Table 3 in this Glossary.

Independently combined probability models. Two probability models are said to be combined independently if the probability of each ordered pair in the combined model equals the product of the original probabilities of the two individual outcomes in the ordered pair.

Integer. A number expressible in the form a or $-a$ for some whole number a .

Interquartile range. A measure of variation in a set of numerical data, the interquartile range is the distance between the first and third quartiles of the data set. Example: For the data set {1, 3, 6, 7, 10, 12, 14, 15, 22, 120}, the interquartile range is $15 - 6 = 9$. See *also*: first quartile, third quartile.

Line plot. A method of visually displaying a distribution of data values where each data value is shown as a dot or mark above a number line. Also known as a dot plot.²

Mean. A measure of center in a set of numerical data, computed by adding the values in a list and then dividing by the number of values in the list. Example: For the data set {1, 3, 6, 7, 10, 12, 14, 15, 22, 120}, the mean is 21.

Mean absolute deviation. A measure of variation in a set of numerical data, computed by adding the distances between each data value and the mean, then dividing by the number of data values. Example: For the data set {2, 3, 6, 7, 10, 12, 14, 15, 22, 120}, the mean absolute deviation is 20.

Median. A measure of center in a set of numerical data. The median of a list of values is the value appearing at the center of a sorted version of the list—or the mean of the two central values, if the list contains an even number of values. Example: For the data set {2, 3, 6, 7, 10, 12, 14, 15, 22, 90}, the median is 11.

Midline. In the graph of a trigonometric function, the horizontal line halfway between its maximum and minimum values.

Multiplication and division within 100. Multiplication or division of two whole numbers with whole number answers, and with product or dividend in the range 0-100. Example: $72 \div 8 = 9$.

Multiplicative inverses. Two numbers whose product is 1 are multiplicative inverses of one another. Example: $3/4$ and $4/3$ are multiplicative inverses of one another because $3/4 \times 4/3 = 4/3 \times 3/4 = 1$.

Number line diagram. A diagram of the number line used to represent numbers and support reasoning about them. In a number line diagram for measurement quantities, the interval from 0 to 1 on the diagram represents the unit of measure for the quantity.

Percent rate of change. A rate of change expressed as a percent. Example: if a population grows from 50 to 55 in a year, it grows by $5/50 = 10\%$ per year.

1. Many different methods for computing quartiles are in use. The method defined here is sometimes called the Moore and McCabe method. See E. Langford, "Quartiles in Elementary Statistics," *Journal of Statistics Education* Volume 14, Number 3 (2006).

2. To be more precise, this defines the *arithmetic* mean.

Probability distribution. The set of possible values of a random variable with a probability assigned to each.

Properties of operations. See Table 3 in this Glossary.

Properties of equality. See Table 4 in this Glossary.

Properties of inequality. See Table 5 in this Glossary.

Properties of operations. See Table 3 in this Glossary.

Probability. A number between 0 and 1 used to quantify likelihood for processes that have uncertain outcomes (such as tossing a coin, selecting a person at random from a group of people, tossing a ball at a target, or testing for a medical condition).

Probability model. A probability model is used to assign probabilities to outcomes of a chance process by examining the nature of the process. The set of all outcomes is called the sample space, and their probabilities sum to 1. *See also:* uniform probability model.

Random variable. An assignment of a numerical value to each outcome in a sample space.

Rational expression. A quotient of two polynomials with a non-zero denominator.

Rational number. A number expressible in the form a/b or $-a/b$ for some fraction a/b . The rational numbers include the integers.

Rectilinear figure. A polygon all angles of which are right angles.

Rigid motion. A transformation of points in space consisting of a sequence of one or more translations, reflections, and/or rotations. Rigid motions are here assumed to preserve distances and angle measures.

Repeating decimal. The decimal form of a rational number. *See also:* terminating decimal.

Sample space. In a probability model for a random process, a list of the individual outcomes that are to be considered.

Scatter plot. A graph in the coordinate plane representing a set of bivariate data. For example, the heights and weights of a group of people could be displayed on a scatter plot.

Similarity transformation. A rigid motion followed by a dilation.

Tape diagram. A drawing that looks like a segment of tape, used to illustrate number relationships. Also known as a strip diagram, bar model, fraction strip, or length model.

Terminating decimal. A decimal is called terminating if its repeating digit is 0.

Third quartile. For a data set with median M , the third quartile is the median of the data values greater than M . Example: For the data set {2, 3, 6, 7, 10, 12, 14, 15, 22, 120}, the third quartile is 15. *See also:* median, first quartile, interquartile range.

Transitivity principle for indirect measurement. If the length of object A is greater than the length of object B, and the length of object B is greater than the length of object C, then the length of object A is greater than the length of object C. This principle applies to measurement of other quantities as well.

Uniform probability model A probability model which assigns equal probability to all outcomes. See *also*: probability model.

Vector. A quantity with magnitude and direction in the plane or in space, defined by an ordered pair or triple of real numbers.

Visual fraction model. A tape diagram, number line diagram, or area model.

Whole numbers. The numbers 0, 1, 2, 3,

Table 1. Common addition and subtraction situations.³

	Result Unknown	Change Unknown	Start Unknown
Add to	Two bunnies sat on the grass. Three more bunnies hopped there. How many bunnies are on the grass now? $2 + 3 = ?$	Two bunnies were sitting on the grass. Some more bunnies hopped there. Then there were five bunnies. How many bunnies hopped over to the first two? $2 + ? = 5$	Some bunnies were sitting on the grass. Three more bunnies hopped there. Then there were five bunnies. How many bunnies were on the grass before? $? + 3 = 5$
Take from	Five apples were on the table. I ate two apples. How many apples are on the table now? $5 - 2 = ?$	Five apples were on the table. I ate some apples. Then there were three apples. How many apples did I eat? $5 - ? = 3$	Some apples were on the table. I ate two apples. Then there were three apples. How many apples were on the table before? $? - 2 = 3$

	Total Unknown	Addend Unknown	Both Addends Unknown ⁴
Put Together/ Take Apart ⁵	Three red apples and two green apples are on the table. How many apples are on the table? $3 + 2 = ?$	Five apples are on the table. Three are red and the rest are green. How many apples are green? $3 + ? = 5, 5 - 3 = ?$	Grandma has five flowers. How many can she put in her red vase and how many in her blue vase? $5 = 0 + 5, 5 = 5 + 0$ $5 = 1 + 4, 5 = 4 + 1$ $5 = 2 + 3, 5 = 3 + 2$

	Difference Unknown	Bigger Unknown	Smaller Unknown
Compare ⁶	(“How many more?” version): Lucy has two apples. Julie has five apples. How many more apples does Julie have than Lucy? (“How many fewer?” version): Lucy has two apples. Julie has five apples. How many fewer apples does Lucy have than Julie? $2 + ? = 5, 5 - 2 = ?$	(Version with “more”): Julie has three more apples than Lucy. Lucy has two apples. How many apples does Julie have? (Version with “fewer”): Lucy has 3 fewer apples than Julie. Lucy has two apples. How many apples does Julie have? $2 + 3 = ?, 3 + 2 = ?$	(Version with “more”): Julie has three more apples than Lucy. Julie has five apples. How many apples does Lucy have? (Version with “fewer”): Lucy has 3 fewer apples than Julie. Julie has five apples. How many apples does Lucy have? $5 - 3 = ?, ? + 3 = 5$

3. Adapted from Boxes 2–4 of *Mathematics Learning in Early Childhood*, National Research Council (2009, pp. 32–33).

4. These take apart situations can be used to show all the decompositions of a given number. The associated equations, which have the total on the left of the equal sign, help children understand that the = sign does not always mean *makes or results in* but always does mean *is the same number as*.

5. Either addend can be unknown, so there are three variations of these problem situations. “Both Addends Unknown” is a productive extension of this basic situation, especially for small numbers less than or equal to 10.

6. For the Bigger Unknown or Smaller Unknown situations, one version directs the correct operation (the version using *more* for the bigger unknown and using *less* for the smaller unknown). The other versions are more difficult.

Table 2. Common multiplication and division situations.⁷

	Unknown Product $3 \times 6 = ?$	Group Size Unknown ("How many in each group?" Division) $3 \times ? = 18$ and $18 \div 3 = ?$	Number of Groups Unknown ("How many groups?" Division) $? \times 6 = 18$ and $18 \div 6 = ?$
Equal Groups	<p>There are 3 bags with 6 plums in each bag. How many plums are there in all?</p> <p><i>Measurement example.</i> You need 3 lengths of string, each 6 inches long. How much string will you need altogether?</p>	<p>If 18 plums are shared equally into 3 bags, then how many plums will be in each bag?</p> <p><i>Measurement example.</i> You have 18 inches of string, which you will cut into 3 equal pieces. How long will each piece of string be?</p>	<p>If 18 plums are to be packed 6 to a bag, then how many bags are needed?</p> <p><i>Measurement example.</i> You have 18 inches of string, which you will cut into pieces that are 6 inches long. How many pieces of string will you have?</p>
Arrays, ⁸ Area ⁹	<p>There are 3 rows of apples with 6 apples in each row. How many apples are there?</p> <p><i>Area example.</i> What is the area of a 3 cm by 6 cm rectangle?</p>	<p>If 18 apples are arranged into 3 equal rows, how many apples will be in each row?</p> <p><i>Area example.</i> A rectangle has area 18 square centimeters. If one side is 3 cm long, how long is a side next to it?</p>	<p>If 18 apples are arranged into equal rows of 6 apples, how many rows will there be?</p> <p><i>Area example.</i> A rectangle has area 18 square centimeters. If one side is 6 cm long, how long is a side next to it?</p>
Compare	<p>A blue hat costs \$6. A red hat costs 3 times as much as the blue hat. How much does the red hat cost?</p> <p><i>Measurement example.</i> A rubber band is 6 cm long. How long will the rubber band be when it is stretched to be 3 times as long?</p>	<p>A red hat costs \$18 and that is 3 times as much as a blue hat costs. How much does a blue hat cost?</p> <p><i>Measurement example.</i> A rubber band is stretched to be 18 cm long and that is 3 times as long as it was at first. How long was the rubber band at first?</p>	<p>A red hat costs \$18 and a blue hat costs \$6. How many times as much does the red hat cost as the blue hat?</p> <p><i>Measurement example.</i> A rubber band was 6 cm long at first. Now it is stretched to be 18 cm long. How many times as long is the rubber band now as it was at first?</p>
General	$a \times b = ?$	$a \times ? = p$ and $p \div a = ?$	$? \times b = p$ and $p \div b = ?$

7. The first examples in each cell are examples of discrete things. These are easier for students and should be given before the measurement examples.

8. The language in the array examples shows the easiest form of array problems. A harder form is to use the terms rows and columns: The apples in the grocery window are in 3 rows and 6 columns. How many apples are in there? Both forms are valuable.

9. Area involves arrays of squares that have been pushed together so that there are no gaps or overlaps, so array problems include these especially important measurement situations.

Table 3. The properties of operations.

Here a , b , and c stand for arbitrary numbers in a given number system. The properties of operations apply to the rational number system, the real number system, and the complex number system.

Associative property of addition	$(a + b) + c = a + (b + c)$
Commutative property of addition	$a + b = b + a$
Additive identity property of 0	$a + 0 = 0 + a = a$
Existence of additive inverses	For every a there exists $-a$ so that $a + (-a) = (-a) + a = 0$.
Associative property of multiplication	$(a \times b) \times c = a \times (b \times c)$
Commutative property of multiplication	$a \times b = b \times a$
Multiplicative identity property of 1	$a \times 1 = 1 \times a = a$
Existence of multiplicative inverses	For every $a \neq 0$ there exists $1/a$ so that $a \times 1/a = 1/a \times a = 1$.
Distributive property of multiplication over addition	$a \times (b + c) = a \times b + a \times c$

Table 4. The properties of equality.

Here a , b , and c stand for arbitrary numbers in the rational, real, or complex number systems.

Reflexive property of equality	$a = a$
Symmetric property of equality	If $a = b$, then $b = a$.
Transitive property of equality	If $a = b$ and $b = c$, then $a = c$.
Addition property of equality	If $a = b$, then $a + c = b + c$.
Subtraction property of equality	If $a = b$, then $a - c = b - c$.
Multiplication property of equality	If $a = b$, then $a \times c = b \times c$.
Division property of equality	If $a = b$ and $c \neq 0$, then $a \div c = b \div c$.
Substitution property of equality	If $a = b$, then b may be substituted for a in any expression containing a .

Table 5. The properties of inequality.

Here a , b , and c stand for arbitrary numbers in the rational or real number systems.

Exactly one of the following is true: $a < b$, $a = b$, $a > b$.

If $a > b$ and $b > c$ then $a > c$.

If $a > b$, then $b < a$.

If $a > b$, then $-a < -b$.

If $a > b$, then $a \pm c > b \pm c$.

If $a > b$ and $c > 0$, then $a \times c > b \times c$.

If $a > b$ and $c < 0$, then $a \times c < b \times c$.

If $a > b$ and $c > 0$, then $a \div c > b \div c$.

If $a > b$ and $c < 0$, then $a \div c < b \div c$.