Embracing Transformational Geometry in CCSS-Mathematics

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Drawing Triangles with a Ruler and Protractor—SAS and ASA

Use a ruler and protractor to draw each triangle. When measurements are requested, give lengths to the nearest 0.1 centimeter and angles to the nearest degree.

Side-Angle-Side Drawings (Given two sides and the angle between them)

For each of these triangles, you're given the measures of two sides and the angle between them.

1. Draw $\triangle PEA$ with $EA = 7$ cm, $m\angle E = 57°$, and $EP = 6$ cm. Measure AP and $\angle A$.
2. Draw $\triangle PAC$ with $m\angle A = 105°$, $AP = 5$ cm, and $AC = 8.5$ cm. Measure PC and $\angle C$.
3. Look at other students' drawings of $\triangle PEA$ and $\triangle PAC$. Do their drawings look like yours? Place your triangles over their triangles, hold them up to the light and see how well the triangles match. If the triangles don't match, recheck each other's measures and see if "human error" might be the reason.

continued

Angle-Side-Angle Drawings (Given two angles and the side between them)

For each of these triangles, you're given the measure of two angles and the side between them.

4. Draw $\triangle NUT$ with $m\angle N = 57°$, $NU = 7$ cm, and $m\angle U = 80°$. Measure NT and TU.

5. Draw $\triangle CAT$ with $m\angle C = 30°$, $CA = 8$ cm, and $m\angle A = 116°$. Measure CT, AT, and $\angle T$.

6. Draw $\triangle MAN$ with $m\angle M = 68°$, $MN = 9$ cm, and $m\angle N = 33°$. Measure MA and NA.

7. Look around at other people's drawings of $\triangle NUT$, $\triangle CAT$ and $\triangle MAN$. Do their drawings look like yours? Place your $\triangle NUT$ over their $\triangle NUT$. Hold them up to the light and see how well the triangles line up. If the triangles don't match, recheck each other's measures for "human error." Do the same for $\triangle CAT$ and $\triangle MAN$.

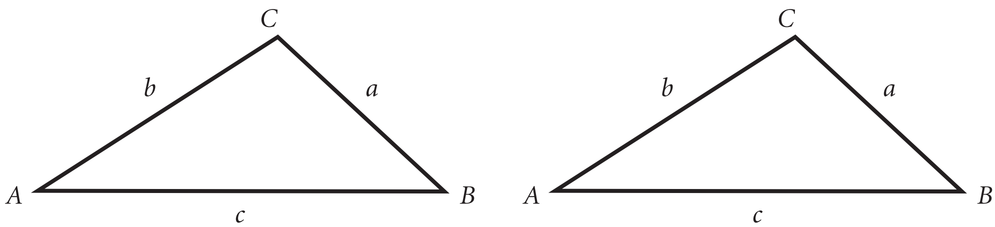
Make Conjectures

8. Do you think that the measures of two sides of a triangle and the angle between them always defines only one triangle?

9. Do you think that the measures of two angles of a triangle and the side between them always defines only one triangle?

What Makes a Triangle?

ASA, SAS, and SSS Triangle Congruence



Two figures are congruent if they can be placed one on top of the other and they match up perfectly. The symbol for congruent is ≅.

Consider this situation: It's the year 1778, and future president George Washington is a surveyor. He is evaluating two triangular plots of land that he thinks may be congruent. But how can he be sure? He can't cut them out and put one on top of the other!

So what does he have to measure to know if the triangles are congruent? Does he have to measure every side and every angle? In fact, there is an easier way. Based on our experiences with drawing triangles in *Drawing Triangles with a Ruler and Protractor—SAS and ASA* and *Drawing Specific Triangles—SSS*, we can introduce three postulates about triangle congruence. These postulates can help the surveyor determine the fewest measurements he can take and yet still be sure the triangles are congruent.

Side-Side-Side (SSS) Triangle Congruence Postulate:

If three sides of one triangle are congruent to the corresponding sides in another triangle, then the two triangles are congruent.

continued

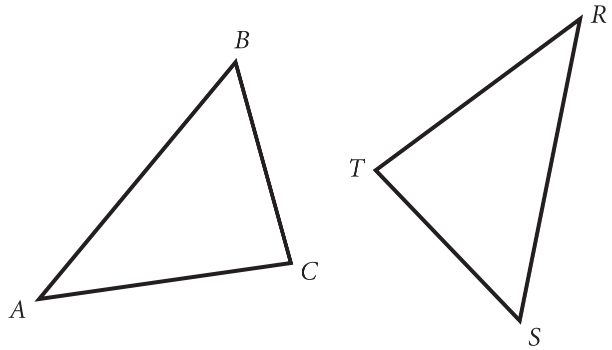
Side-Angle-Side (SAS) Triangle Congruence Postulate:

If two sides and the included angle in one triangle are congruent to the corresponding sides and angle in another triangle, then the two triangles are congruent.

Angle-Side-Angle (ASA) Triangle Congruence Postulate:

If two angles and the included side in one triangle are congruent to the corresponding angles and side in another triangle, then the two triangles are congruent.

George sketched the two plots he was evaluating and labeled them.



1. If George measures that one triangular plot has $m\angle A = 43°$, $AC = 3.62$ mi, and $m\angle C = 82°$, and the other triangular plot has $m\angle R = 43°$, $RT = 3.62$ mi, and $m\angle T = 82°$, can he be sure the two plots are congruent? If so, according to what triangle congruence postulate?

2. If George measures that one triangular plot has $BC = 3.2$ miles $AC = 4.9$ miles, and $m\angle C = 78°$, and the other triangular plot has $RT = 4.9$ miles, $m\angle T = 78°$, and $ST = 3.2$ miles, can he be sure the two plots are congruent? If so, according to what triangle congruence postulate?

continued

Tricky Triangles

The triangle congruence postulates can be used to determine if two triangles are congruent. But they can also be used to determine if a set of measures determines only one triangle. For example, if you're given three sides of a triangle, does that determine only one possible triangle? It does, because any other triangle with the same three sides would be a congruent triangle.

Mr. Arnold Buford makes up a set of problems specifying triangles for his students to draw. To give his students something to argue about, he includes some instructions that won't work.

Try to draw each triangle below. Decide if a unique triangle is determined. If the instructions do determine a unique triangle, state which triangle congruence postulate applies. If the instructions do not determine a unique triangle, explain why not.

3. $\triangle ABC$ with $m\angle A = 88°$, $AB = 32$ cm, and $m\angle B = 110°$
4. $\triangle SAP$ with $m\angle P = 45°$ and $SA = 12$ cm
5. $\triangle PEA$ with $m\angle P = 52°$, $m\angle E = 105°$, and $m\angle A = 23°$
6. $\triangle NUT$ with $m\angle N = 105°$, $NU = 5$ cm, and $NT = 6.6$ cm
7. $\triangle QED$ with $QE = 4$ miles, $QD = 7$ miles, and $ED = 2$ miles
8. $\triangle PAL$ with $PA = 10$ cm and $AL = 4$ cm
9. $\triangle END$ with $EN = 12$ cm, $ED = 6.3$ cm, and $ND = 5.9$ cm

Walking Sets of Equidistant Points

How many points make up a line?

If you said "infinite," you are right! For this activity, you'll need to think of a line as being made up of an infinite set of points.

1. For each description in "The set of points **equidistant** from" column, draw a sketch that helps you decide which of these names describes the set:

   Angle bisector   Parabola   Circle   Perpendicular bisector   Parallel line

The set of points equidistant from...	Is a...
A given point	
The endpoints of a given line segment	
A given point and a given line	
Two parallel lines	
Two intersecting lines	

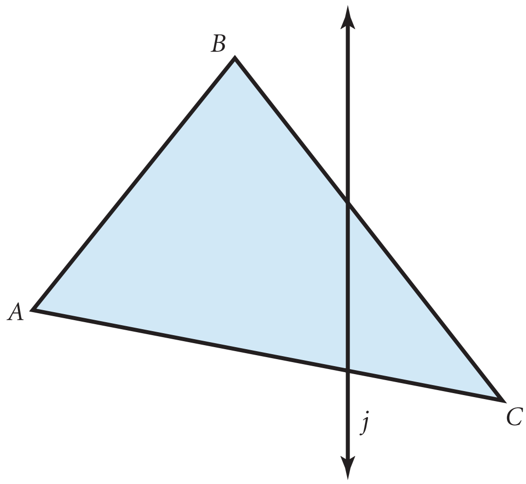
2. Discuss how to model each description in the left column using group members. Have one person walk a path that traces out the set of points. You may use string as a prop if needed.

   For example here is how one group decided to model "The set of all points equidistant from the endpoints of a given line segment": Arnold and Livvy stand about 10 feet apart to serve as endpoints of a line segment. Imani stands between them at a point equidistant from both endpoints. Jackson asks: Where else can Imani move and still be equidistant from Arnold and Livvy? Then Imani walks that path.

Reflection Challenges

Reflect a Triangle

1. Draw a triangle about the size of your palm and label the vertices A, B, and C.
2. Draw a line j that passes through two sides of the triangle.



3. Find the most efficient way to use a compass and straightedge to construct the reflection of $\triangle ABC$ across line j.

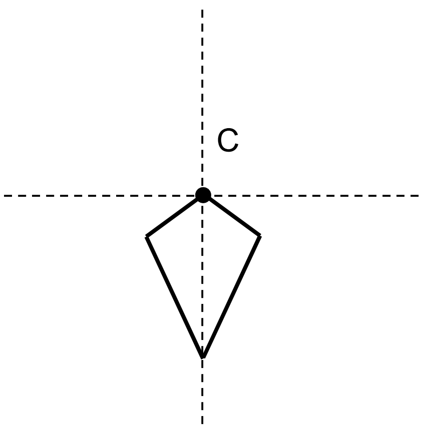
continued

ROTATIONS

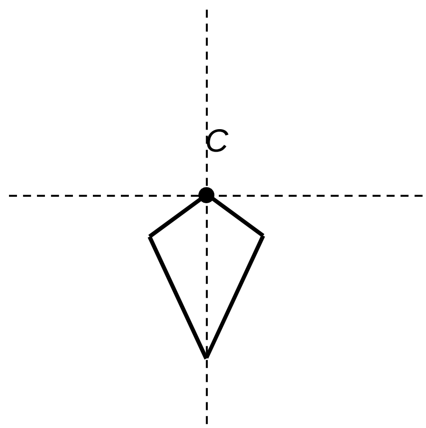
A <u>rotation</u> about a point *P* through angle α is a transformation such that: (1) if point *A* is different from *P*, then $PA = PA'$ and the measure of $\angle APA' = \alpha$; and (2) if point *A* is the same as *P*, then $A' = A$.

- Draw this shape on patty paper with pencil markings on both sides of the paper.
- For each figure and point of rotation *C*, visualize where the image will be.
- Perform the transformation using patty paper and record.
- Shade the original figure blue and its image red.

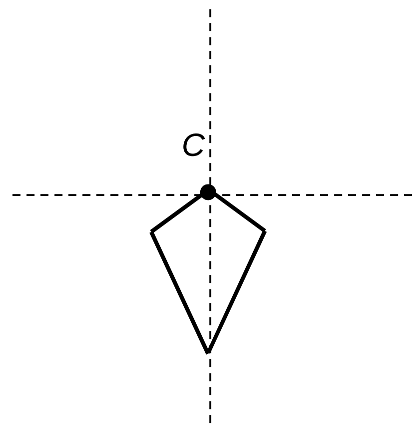
1a. Rotate 90°.



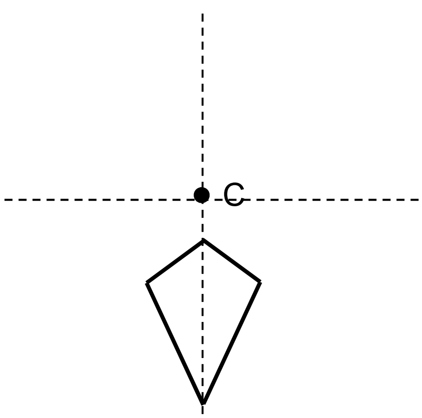
1b. Rotate 180°.



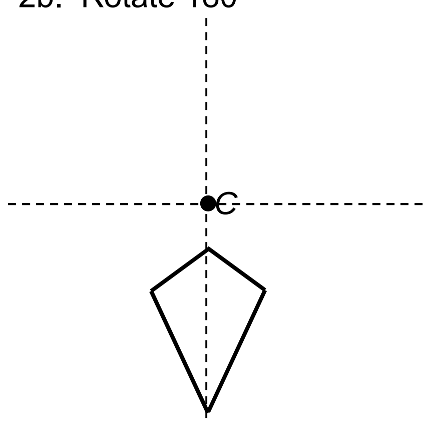
1c. Rotate 270°



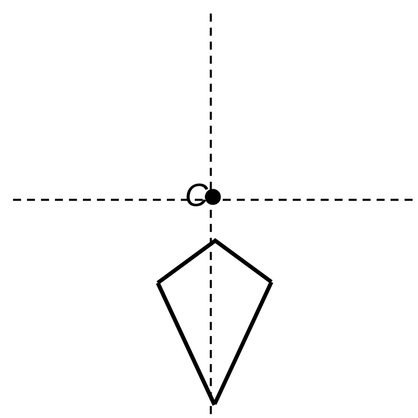
2a. Rotate 90°.



2b. Rotate 180°



2c. Rotate 270°

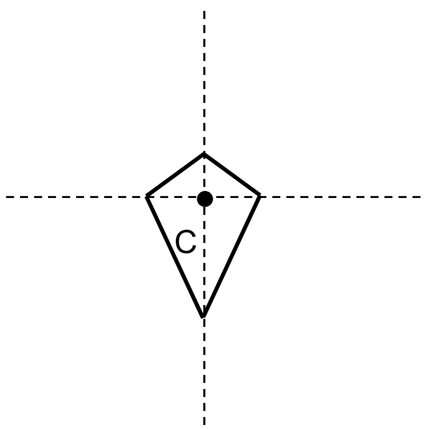


3. Make a conjecture about the relationship between the distance from the center *C* of the rotation to corresponding points on the figure and its image.

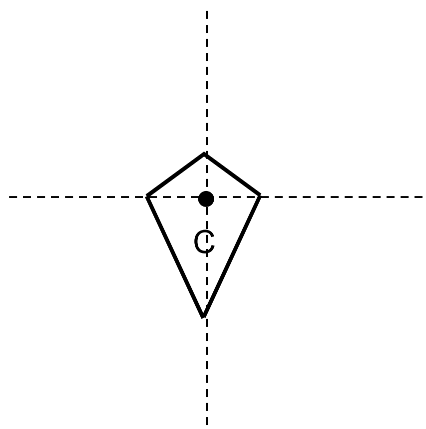
ROTATIONS (continued)

- Draw this shape on patty paper with pencil markings on both sides of the paper.
- Shade each figure blue.
- For each figure and point of rotation *C*, visualize where the image will be.
- Perform the transformation using patty paper and record.
- Shade the original figure blue and its image red.

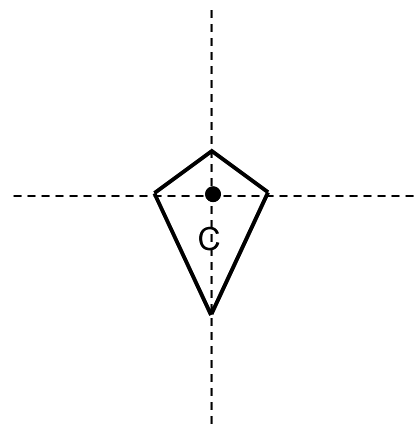
4a. Rotate 90^{o}.



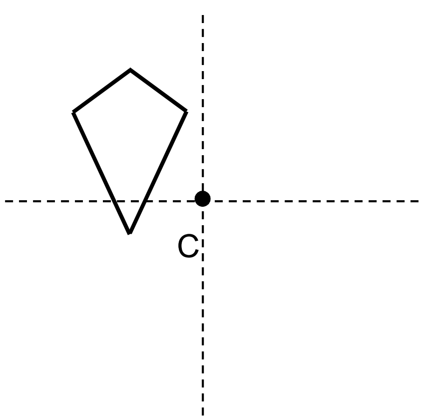
4b. Rotate 180^{o}.



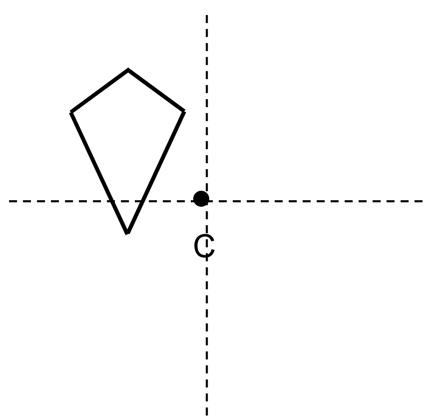
4c. Rotate 270^{o}.



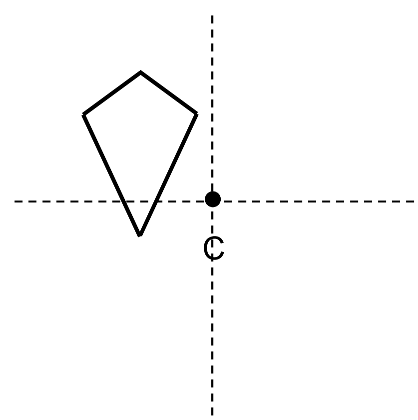
5a. Rotate 90^{o}.



5b. Rotate 180^{o}.



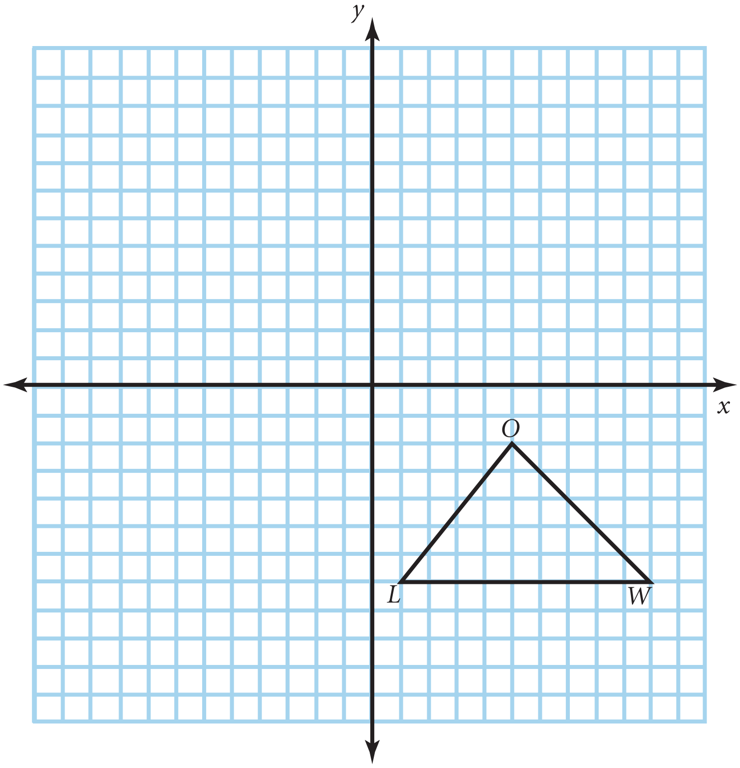
5c. Rotate 270^{o}.



6. Suppose the letter “Q” were placed somewhere on this paper. What would the image of Q be under the rotation for 5B?

7. Does your conjecture about distances from problem 3 on the previous page hold for these examples? Revise your conjecture as needed.

Rotation with Coordinates



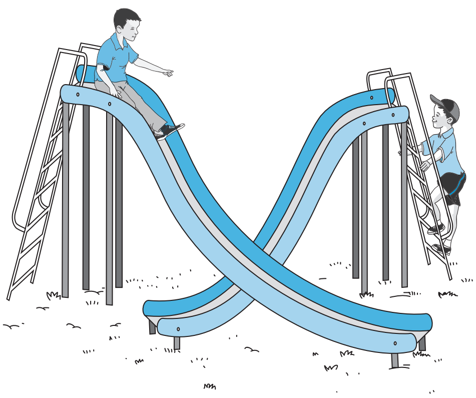
Triangle *LOW* has vertices at $L(1, -7)$, $O(5, -2)$, and $W(10, -7)$.

1. Draw the image of $\triangle LOW$ when rotated 90° with center of rotation (0, 0). What are the coordinates of the vertices of the image triangle?

2. Suppose point (p, q) is rotated 90° around (0, 0). What are the coordinates of the image of (p, q)?

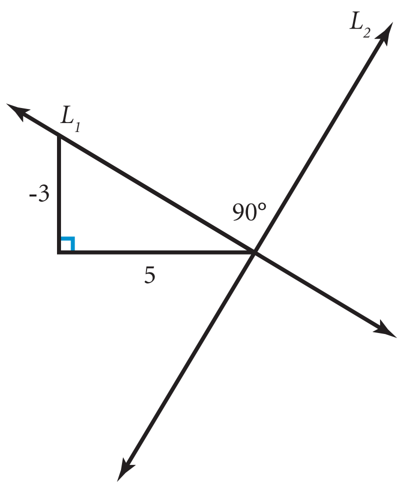
Sloping Slides

Ella Pea works at the Exploratorium, an active science museum for children. She is designing a new exhibit involving slides. She wants to build two slides next to each other, but facing opposite directions. One slide will be steep and the other will have a gentler slope. The slides are to be perpendicular to each other at the point where they pass.



If the sliders time their starts right, they can high-five each other at the point where the slides pass.

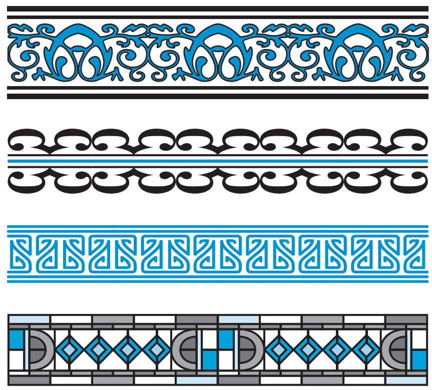
Ella wants the gentler slide to have a slope ratio of $-\frac{3}{5}$. That is, for every 5 feet of horizontal movement, the slide will drop 3 feet. The design can be modeled with this diagram:



1. What slope should the steeper slide have so that the slides are perpendicular?
2. Solve this problem for any given slope. In other words, instead of a slope ratio of $-\frac{3}{5}$, suppose the gentler slide has slope m. What must be the slope of the steeper slide in terms of m?

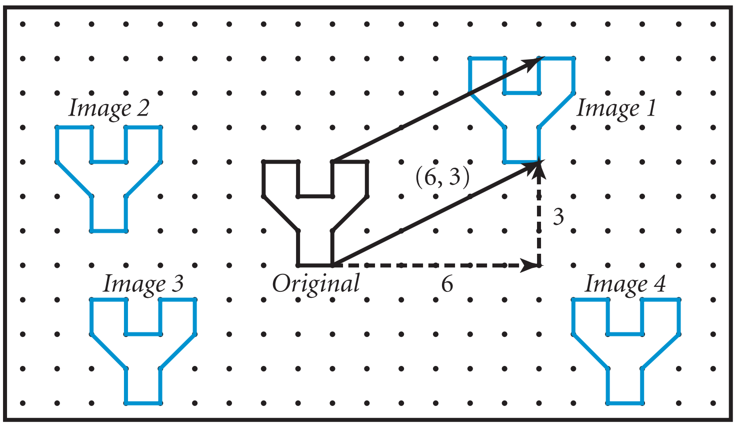
Isometric Transformation 3: Translation

These are medieval frieze designs. Each consists of a design repeated over and over along a line. The basic shape is *translated* along the line.



In a **translation,** a figure is moved along a straight line without any rotation. Translation is another isometric transformation. In the frieze strips shown, the translations are all horizontal, but a translation can go in any direction.

Often a translation is broken into two parts: a horizontal shift (left or right) and a vertical shift (up or down).



In the diagram above, the original Y-shape was moved to Image 1 by a horizontal shift of 6 units and a vertical shift of 3 units. This combined movement can be written as the *vector* $\langle 6, 3\rangle$, indicating a shift of 6 in the x-direction and 3 in the y-direction.

The translation $\langle 6, 3\rangle$ moves every point of the original Y to a corresponding point on Image 1.

Verify that the translation that moves the original to Image 2 is $\langle -6, 1\rangle$.

1. What translation moves the original to Image 3?
2. What translation moves the original to Image 4?

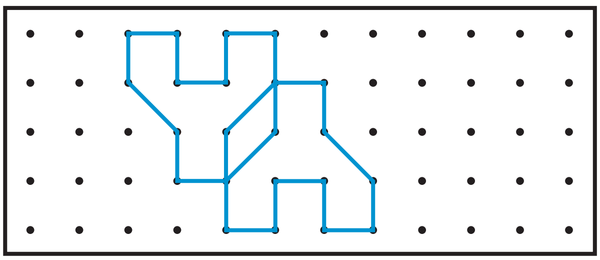
Frieze Frame

Look back at the medieval friezes shown in *Isometric Transformation 3: Translation*. Each frieze starts with a basic design, which is translated along a line to make images of the original. The basic designs are often made up of transformations of an even simpler shape.

In this activity, you'll create your own frieze design using square dot paper.

1. Decide on a simple shape. On dot paper, make a shape that includes some shading and fills up a 3-by-3 grid like the "Y" on the previous page. Your simple shape can be anything you like, or might be based on your first initial.
2. Make a basic design using your simple shape, using at least one reflection or rotation. Your basic design should fit within a 6-by-9 grid.

   For example, the "Y" could be used to make a basic design by adding an image of a 180° rotation. This design fits inside a 4-by-5 grid.



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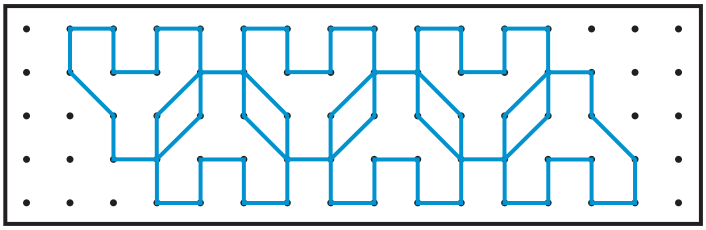
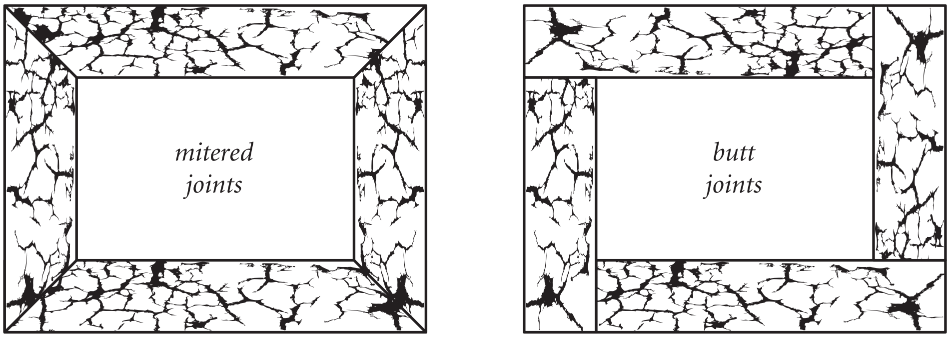
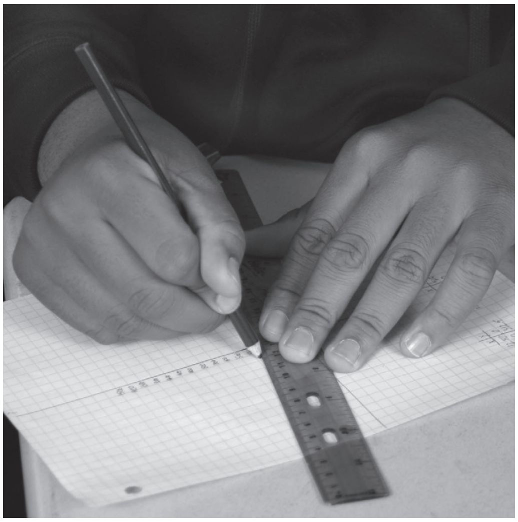
3. Make a frieze consisting of at least three translated copies of your basic design.



4. Use your frieze as a border to draw a rectangular picture frame. Most frames use mitered joints at the corners, as shown below. Mitered joints are more elegant than butt joints. For mitered joints you may have to figure out how to make a 45° cut through your basic design so the ends will match up nicely.



Translation Investigations



1. Translating Coordinates

What happens to a point on coordinate axes when it is translated? That is, when point (p, q) is translated by the vector $\langle h, k \rangle$, what are the coordinates of the image of the point? Investigate by trying out some translations of points on graph paper.

2. Translating Lines

In the *Fireworks* unit in Algebra 1, you learned how to move the vertex of a parabola to a new location by making changes in the parabola's equation. Now you'll figure out how to translate a line by changing its equation.

Start with a line that is easy to graph, like $y = 2x + 3$. First figure out what happens to the equation when the line is translated vertically. Next figure out what happens to the equation when the line is translated horizontally. Finally, put the ideas together. What is the equation when the general line $y = mx + b$ is translated by the vector $\langle h, k \rangle$?

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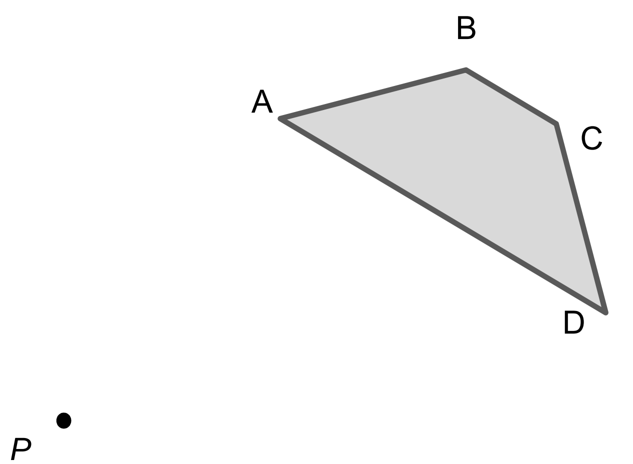
INTRODUCTION TO DILATIONS

A **dilation** with center P and scale factor $k \neq 0$ is a mapping such that if point A is different from P, then the image A' lies on line PA and $PA' = |k| \cdot PA$

1. Use two small rubber bands. Link them together. Anchor one end of the band at point P with a pencil. Put another pencil into the other end of the second band. Move the pencil so that the knot linking the bands traces over the figure. You will create a new figure that is (approximately) a dilation of the first.



2. Color the original figure blue. Color the image red. Label it *A'B'C'D'*. Be sure to mark corresponding points with the same letter.

3. Draw line segments connecting point P to corresponding points. Find these ratios:

$\frac{PA'}{PA} =$ $\frac{PB'}{PB'} =$ $\frac{PC'}{PC} =$ $\frac{PD'}{PD} =$

4. What do you notice about these ratios?______________________________

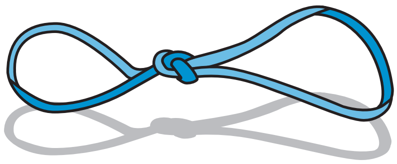
5. What is the scale factor for this dilation?___________

Dilation with Rubber Bands

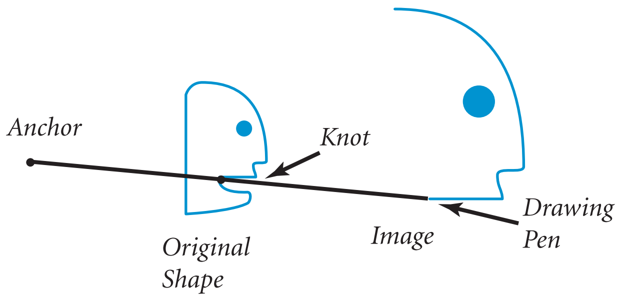
For this activity, you will need:

- two rubber bands
- two ballpoint pens

1. Loop two identical rubber bands together as shown to knot them together in the middle:



2. On a sheet of paper, draw a shape to be enlarged, and put a dot outside it that will serve as a center of dilation.
3. Hold the tip of one ballpoint pen on the center of dilation, with one end of the rubber band looped around it. Put the other pen inside the end of the other rubber band.
4. Stretch the rubber band and trace the knot along your shape while drawing with the second pen.



5. Does your result look like a dilation of the original shape? What property of rubber bands makes this method of dilation possible?
6. How could you use rubber bands to make an image that is three times as big as the original?
7. How could you use rubber bands to make an image that is 1.5 times as big as the original?

Dilating a Right Triangle

1. Draw right triangle ABC with legs $AB = 3.2$ cm and $AC = 4.5$ cm.
2. Locate a center of dilation inside the triangle roughly equidistant from the vertices. Draw projection lines from the center of dilation, one through each vertex.
3. Measure along the projection lines with your compass to locate the vertices of an image triangle dilated to twice the size of the original. Draw the image.
4. Use the same projection lines to make an enlargement 2.5 times as big as the original.
5. Measure the hypotenuse of the original and each image. Use the measurements to calculate the scale factor for each hypotenuse. How accurate are your results?

Enlarging on a Copy Machine

Arnold's sister has painted a beautiful logo for him. But Arnold needs to make the logo three times its original size so it will fit the space allotted on his advertising poster. His office has a copier, but the biggest enlargement it will do is 200%.

Arnold asks his friends: "How can I use this copier to make a 300% enlargement?"

Everyone likes to give advice and Arnold's friends are not exceptions. Which of these suggestions will actually make a 300% enlargement of the original logo? For those that don't make a 300% enlargement, what enlargement do they actually make?

Trudy says:

> "I would do it in three steps. First make a 100% enlargement. Then take that image and make another 100% enlargement. Finally, one more 100% enlargement of that image will make a logo three times as large."

Lizette says:

> "You only need two steps. First make a 150% enlargement. Then take that enlargement and make a 150% enlargement of it. The result will be a three times enlargement."

continued

Jackson says:

"First make a 200% enlargement of the original. Then put that image on the machine and make a 150% enlargement. The result will be a 3 times enlargement."

Kayla says:

"First do a 200% enlargement. Then take the image and enlarge it another 200%. Finally take that image and shrink it 75%. Result: Triple the size of the original!"

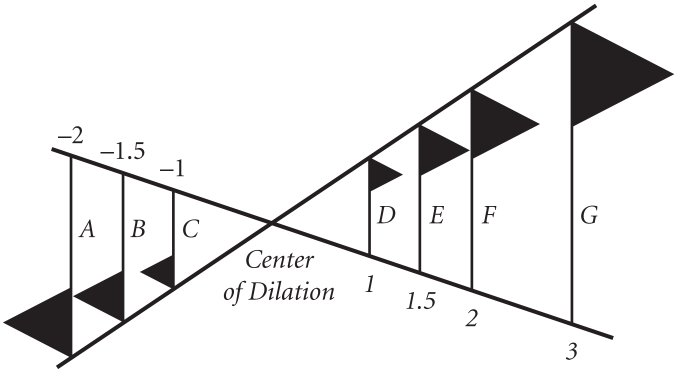
Madelyn says:

"Well, the square root of 3 is about 1.73, so I'd set the copier at 173%. Then I'd make the first enlargement, and then, keeping the same setting, make an enlargement of the enlargement. That will give a 300% enlargement."

Dilation Investigations

Repeated Dilations

This diagram shows dilations of a pennant, with center of dilation and projection lines drawn.

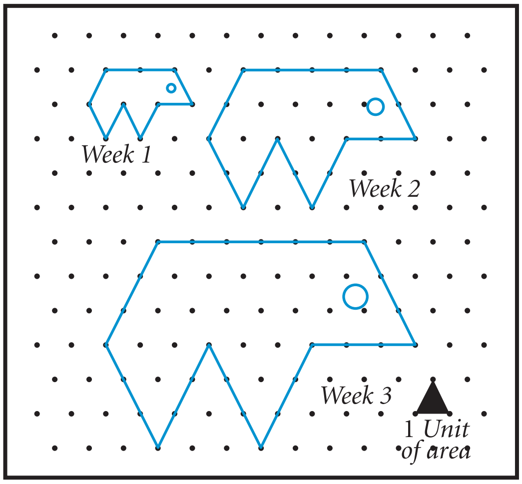


1. What scale factor of image *D* gives *G* as the image?
2. What scale factor of image *F* gives *G* as the image?
3. What scale factor of image *G* gives *F* as the image?
4. What scale factor of image *A* gives *G* as the image?
5. What is the image when *F* is dilated by scale factor of −1?
6. What sequence of dilations will take *D* onto *F* and then take *F* onto *A*?
7. If image *C* is dilated by a factor of 2 and the resulting image is dilated by a factor of −1.5, which pennant is the final image?
8. What sequence of dilations will take image *D* onto *B* and then take *B* onto *G*?
9. True or False?: The result of two positive dilations is always a positive dilation.
10. True or False?: The result of two negative dilations is always a negative dilation.

continued

Billy Bear Grows Up

Arnold takes care of Billy Bear at the Dotty Zoo. Billy Bear is growing up fast. Just look at how quickly he has grown over the first three weeks!



By Week 2, Billy grew by a scale factor of 2. By week 3, Billy had increased his size by a scale factor of 3. Assume that this trend continues: In Week N, Billy has increased his size from the original by a scale factor of N.

11. On isometric dot paper, draw Billy Bear at Week 4.

12. Copy and complete the table below with the perimeter and area of Billy Bear at each week. Arnold put in data for the first two weeks. Draw more images of Billy if needed.

Weeks	1	2	3	4	5	6	7	8
Perimeter Units	9	18						
Area Units	7	28						

13. If Billy keeps growing like this, what will his perimeter and area be after 20 weeks?

14. If Billy grows like this for N weeks, what will his perimeter and area be?

Properties of Euclidean Transformations

A <u>transformation</u> of the plane is a one-to-one mapping (function) of the plane onto itself.

We have explored 4 types of Euclidean transformations. Please complete the table for the 4 transformations.

Under the translation:	True(T) or False(F)			
	Reflection	Rotation	Translation	Dilation
a. **<u>Distance</u>** is preserved. In other words, lines are taken to lines, and line segments to line segments of the same length.				
b. **<u>Parallelism</u>** is preserved. In other words, parallel lines are taken to parallel lines.				
c. **<u>Angle measure</u>** is preserved. In other words, angles are taken to angles of the same measure.				
d. **<u>Collinearity</u>** is preserved. In other words, if three points lie on the same line, then their images lie on the same line.				
e. **<u>Betweeness</u>** is preserved. In other words, if B is between A and C on a line, then B' is between A' and C' on the image.				

Any translation which preserves distance, parallelism, angle measure, collinearity, and betweeness is called an <u>isometry</u>.

Based on your observations, which transformations are also isometries? Justify your answer!